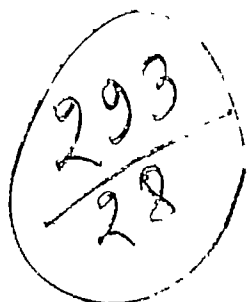


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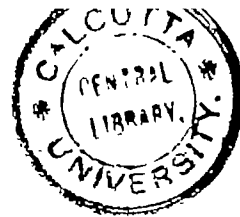
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ON AN ALGEBRAIC SYSTEM GENERATED BY A SINGLE ELEMENT AND ITS APPLICATION IN RIEMANNIAN GEOMETRY

By

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(Received November 4, 1949)

1. In the course of investigation on parallel displacement in Riemannian space, a certain system of affine connections was obtained which appeared to behave in an interesting manner. This system is given in the last article. It was thought natural to enquire whether there existed an abstract algebraic system which would cover the system mentioned above. It was, however, apparent at the outset that the well-known algebraic systems, such as ring, field, group etc., did not satisfy the requirements, and that a new (non-associative) system had to be created. This new system is discussed in this paper; and instead of beginning with a formal definition of the system all at once, it seems desirable to describe it gradually.

Let us then start with an abstract element a and suppose that corresponding to a , there exist two other elements, called the *associate* and the *conjugate* of a and denoted by a^* and a' respectively. Suppose further that there exist elements which are associate and conjugate of a^* and of a' and of every element generated in this way, and that this process can be continued indefinitely. Let the set of all elements thus obtained be denoted by T . The associate and the conjugate shall be governed by the property that if t is an arbitrary element of T , the associate of the associate as well as the conjugate of the conjugate of t is t itself, written

$$t^{**} = t, \quad t'' = t. \quad (1.1)$$

It thus follows that the property of being associate is mutual or symmetric and the same thing holds for conjugacy. Let us imagine that there are *operations* by which the associate and the conjugate of an element are obtained. These operations, except with regard to the property (1.1), are however quite arbitrary; but once chosen they must remain unaltered in a particular investigation.

In adopting the notation for the repeated use of these operations, we shall suppose that the symbols $*$, $'$, which we shall speak of as *suffixes* attached to an element, are written from the left to the right in order of priority of application; e.g., $a^{*'}$ and a'^{*} shall denote respectively the conjugate of the associate and the associate of the conjugate of a . We may therefore notice that the operations with suffixes satisfy the associative law. Thus, for example (as in the case of composition with the inverse element of a 'group'),

$$a^{*'*'} = a'^{***} = a'^{*''} = a'^{*}, \quad a'^{**'*'} = a.$$

An element which is its own associate will be called a *self-associate* element; similarly for a *self-conjugate* element. Obviously, if a is both self-associate and self-conjugate, T consists of just one element; and if the operations for obtaining the associate and the conjugate are not distinct from one another, T consists of not more than two elements. These are trivial cases, and we shall exclude both these possibilities.

2. The elements of T may be arranged as forming the following two sequences:

$$\begin{aligned} S_1: & a, a^*, a^{*'}, a^{**}, a^{**'}, \dots \} \\ S_2: & a', a'^*, a'^{*'}, a'^{**}, a'^{**'}, \dots \} \end{aligned} \quad (2.1)$$

Obviously, the r -th term of S_1 has $r-1$ suffixes and of S_2 r suffixes. Consider the possibility of two terms of one of the sequences being equal to one another. Let $t_1, t_2, t_3, \dots; s_1, s_2, s_3, \dots$ denote the successive terms of S_1 and S_2 respectively.

I. Let t_{p+1} be the first (earliest) term which is equal to a preceding term, say t_q ; $p, q = 1, 2, \dots; q < p+1$. Four cases may arise:

(i) p even, q odd. Here

$$a^{**' \dots'} \text{ with } p \text{ suffixes} = a^{**' \dots'} \text{ with } q-1 \text{ suffixes.}$$

Taking successively the conjugate and the associate of both sides,

$$a = a^{**' \dots'} \text{ with } p-q+1 \text{ suffixes, or, } t_1 = t_{p-q+1}.$$

By hypothesis, $q < p+1 \leq p-q+2$; therefore $q = 1$. And, as the suffixes of t_{p+1} end in a dash, we have

$$t_r = t_{p+r}, \quad r = 1, 2, \dots, \text{ i.e., } r = 1, 2, \dots, p \pmod{p}. \quad (2.2)$$

Hence the sequence S_1 becomes a *cyclic* sequence with p (even) distinct terms. Again it follows from above that

$$\left. \begin{aligned} a &= a^{**' \dots'} = a'^{*'} \dots * \text{ with } p \text{ suffixes,} \\ \therefore a^{**' \dots'} &= a'^{*'} \dots * \text{ with } p/2 \text{ suffixes.} \end{aligned} \right\} \quad (2.3)$$

It is thus seen that S_2 is S_1 with the opposite sense, i.e.,

$$t_r = s_{p+1-r}, \quad s_r = t_{p+1-r}, \quad r = 1, 2, \dots, p. \quad (2.4)$$

(ii) p, q both even. Here, proceeding as before,

$$a = a'^{*'} \dots * \text{ with } p+q-1 \text{ suffixes, or, } t_1 = t_{p+q}.$$

Taking successively the associate and the conjugate of both sides,

$$t_r = t_{p+q-r+1}, \quad r = 1, 2, \dots, p+1.$$

Thus,

$$t_p = t_{q+1}, \quad t_{p+1} = t_q.$$

By hypothesis

$$\begin{aligned} q+1 &\nless p+1, \quad q < p+1, \quad \therefore q = p \\ \therefore t_{p+1} &= t_p. \end{aligned}$$

As the suffixes of t_p end in a star, t_p is self-conjugate. It follows that in this case

$$\begin{aligned} t_{p+r} &= t_{p-r+1}, \quad r = 1, 2, \dots, p \\ \therefore t_{2p} &= t_1. \end{aligned}$$

As the suffixes of t_{2p} end in a star, we have the following result:

$$t_{p+r} = t_{p-r+1}, \quad r = 1, 2, \dots, p; \quad t_{2p+r} = s_r, \quad r = 1, 2, \dots \quad (2.5)$$

It is thus seen that the second half of the first $2p$ terms of S_1 is the reverse of the first half, after which S_2 follows. S_1 is therefore an infinite sequence which includes S_2 .

(iii) p, q both odd. This case is similar to case (ii). Here also $t_{p+1} = t_p$. But as the suffixes of t_p now end in a dash, t_p is self-associate, and the terms are then reversed up to t_{2p} , after which the (infinite) sequence S_2 follows.

(iv) p odd, q even. Here, as in (i), $t_1 = t_{p-q+2}$. Therefore, $q = 1$ contrary to the supposition that q is even. Hence this case cannot arise.

II. Let s_{p+1} be the first (earliest) term which is equal to a preceding term, say s_q . Here, as in I, we have four cases:

(i) p even, q odd. Proceeding as before, it is seen that $s_1 = s_{p-q+2}$. So this case is exactly similar to I(i). S_2 is a cyclic sequence with p terms and S_1 is the reverse of it.

(ii) p, q both even. Here also we obtain result similar to that in I(ii). s_p is self-conjugate, after which the terms are reversed up to s_{2p} and then follows S_1 . S_2 is therefore an infinite sequence which contains S_1 .

(iii) p, q both odd. Here too the result is similar to that in I(iii). s_p is now self-associate.

(iv) p odd, q even. As in I(iv), this case cannot arise.

All these properties are repeated if S_1 and S_2 are replaced by the sequences

$$a, a', a'^*, a'^{**}, \dots \text{ and } a^*, a^{*'}, a^{*'*}, a^{*'*'}, \dots$$

Examples of the cases I(i), II(i).

$$p = 2: \quad a^* = -a \quad \text{or} \quad 1/a \quad \text{or} \quad -1/a, \quad a' = a,$$

where $a \neq 0$ is an arbitrary element of a field.

$$p = 4: \quad (1) \quad a^*, a' = -a, \pm c/a,$$

where a is a number, real or complex, c is a constant number and $ac \neq 0$.

$$(2) \quad a^*, a' = M^T, M^{-1},$$

where M is a square matrix having non-zero determinant and M^T, M^{-1} are respectively its transposed and inverse.

3. The problem with which the present paper deals is to enquire whether it is possible to obtain, with the help of the elements of T , a new element which shall be both self-associate and self-conjugate. Obviously, the problem becomes less complicated when we have to deal with a finite T . And T is finite only in the cases I(i), II(i) of the last article. We shall be concerned in this paper with a finite T , i.e., with a finite cyclic sequence (as in the case of a finite cyclic 'group') and, as such, there is no necessity to distinguish between the sequences S_1 and S_2 . Take one of the sequences, say S_1 , the terms of which now form a finite set of order p (even), and in which every term is of order p (in the sense that $t = t^{**} \dots = t^{*'} \dots$, with p suffixes). In what follows it will be advantageous to speak of the terms or elements of S_1 as "digits". These digits will,

as before, be denoted by t_1, t_2, \dots, t_p , and they will be supposed to be all distinct from one another.

In order to obtain a new element having the proposed property, we have to construe a system of elements containing this particular element. Naturally therefore, every element of such a system shall have an associate and a conjugate, the operations for which shall be the same as those for the digits. That is to say, if the symbols h and g are used to denote these operations by writing

$$t^* = h(t), \quad t' = g(t),$$

where t is an arbitrary digit, then for an arbitrary element x of the proposed system, we shall have

$$x^* = h(x), \quad x' = g(x).$$

And, as $t^{**} = t$, the symbol h must satisfy

$$h(x^*) = h(h(x)) = [h(x)]^* = x.$$

Similarly for the symbol g .

Let us then introduce a *composition* for every pair of digits, whereby two digits t_i, t_j are composed to form an element $t_i \circ t_j$, and we suppose that the composition is commutative.

Let us further suppose that for every such element $t_i \circ t_j$, there exist its associate $(t_i \circ t_j)^*$ and conjugate $(t_i \circ t_j)'$, the operations for the formation of which being the same as for the digits, as explained above. And for these operations the composition shall satisfy the following conditions:

$$(t_i \circ t_j)^* = t_i^* \circ t_j^*, \quad (t_i \circ t_j)' = t_i' \circ t_j' \quad (3.1)$$

Subject to the above conditions, the composition can be chosen (if at all) arbitrarily; but when the choice has once been made, it must remain unaltered for a particular investigation.

Let A be the set of all elements $t_i \circ t_j$. Then A has the property that it is finite and that the associate and the conjugate of every element of A belong to A .

Now, using the same composition and the same operations for the formation of associates and conjugates, let it be possible to *extend* the set A to a system S having the following properties:

- (1) if ξ and η are two elements of S , $\xi \circ \eta$ is an element of S ;
- (2) every element of S has an associate and a conjugate belonging to S ;
- (3) for every pair of elements ξ, η of S , the composition satisfies the conditions

$$(\xi \circ \eta)^* = \xi^* \circ \eta^*, \quad (\xi \circ \eta)' = \xi' \circ \eta'. \quad (3.2)$$

This system S is the required system and it is generated by the single element a .

We shall assume, in order to avoid initial complications, that S does not contain more than one element which is both self-associate and self-conjugate. The justification for this assumption lies in the fact that there exists system which has this property, *e.g.*, the system given in the last article. The necessary and sufficient condition for the existence of one or more such elements requires investigation.

It follows from the assumption that the set A does not contain more than one element which is both self-associate and self-conjugate. For, if possible, let A contain two such elements

$$u_1 = t_1 \circ t_j, \quad u_2 = t_k \circ t_l.$$

Then it follows from (3.1) and (3.2) that

$$u_3 = u_1 \circ u_2$$

is an element of S which is both self-associate and self-conjugate. Now $u_3 \neq u_1$; for, if $u_3 = u_1$, then, by symmetry $u_3 = u_2$, and therefore $u_1 = u_2$, contrary to the hypothesis. As a matter of fact, if A contains more than one element which is both self-associate and self-conjugate, then the system S contains a sub-system of infinite order, each element of which is both self-associate and self-conjugate.

4. We now establish some formulae for the construction of the element of S which is both self-associate and self-conjugate when p has special values and the composition is of special nature. Consider the following particular cases:

If $p = 2$, then $a^* = a'$. Therefore $a \circ a^*$ is both self-associate and self-conjugate. Conversely, if $a \circ a^*$, which is obviously self-associate, is also self-conjugate, then $a \circ a^* = a' \circ a'^*$. This is satisfied if $a^* = a'$, i.e., if $p = 2$. Thus

$$t_1 \circ t_2$$

is both self-associate and self-conjugate if and only if $p = 2$.

If $p = 4$, then $a^{**} = a'^*$. Therefore $a \circ a^{**} = a \circ a'^* = u$, say. Whence follows $u^* = u'$. Accordingly, by the previous case, $u \circ u^* = (a \circ a^{**}) \circ (a^* \circ a'^*)$ is both self-associate and self-conjugate. Conversely, if $u \circ u^*$, which is obviously self-associate, is also self-conjugate, then $u \circ u^* = u' \circ u'^*$. This is satisfied if $a^{**} = a'^*$, i.e., if $p = 4$. Hence

$$(t_1 \circ t_3) \circ (t_2 \circ t_4)$$

is both self-associate and self-conjugate if and only if $p = 4$.

If $p = 8$, then $a^{***} = a'^{**}$. Therefore $a \circ a^{***} = a \circ a'^{**} = v$, say. Whence follows $v^{**} = v'^*$. Accordingly, by the previous case, $(v \circ v^{**}) \circ (v^* \circ v'^*)$ is both self-associate and self-conjugate. Conversely, if this element is both self-associate and self-conjugate, then, as in the previous case, $p = 8$. Hence

$$((t_1 \circ t_3) \circ (t_3 \circ t_7)) \circ ((t_2 \circ t_4) \circ (t_4 \circ t_8))$$

is both self-associate and self-conjugate if and only if $p = 8$.

The general result follows by mathematical induction, namely

$$((\dots ((t_1 \circ t_{1+2^{r-1}}) \circ (t_{1+2^{r-1}} \circ t_{1+2^{r-1}+2^{r-1}})) \circ ((t_{1+2^{r-1}} \circ t_{1+2^{r-1}+2^{r-1}}) \circ \dots \\ \dots ((t_{2^{r-1}} \circ t_{2^{r-1}+2^{r-1}}) \circ (t_{2^{r-1}} \circ t_{2^r})) \dots)) \quad (4.1)$$

is both self-associate and self-conjugate if and only if $p = 2^r$.

Suppose now as a special case that our composition $t_i \circ t_j$ can be defined in the following way:

For a certain kind of composition $\lambda = t_i \circ t_j$ for every pair of digits, let there exist elements every one of which is a function $f(\lambda)$ of λ . The word 'function' is used here in the sense that there is a one-to-one correspondence $\lambda \leftrightarrow f(\lambda)$ between the composition of every pair of digits λ and $f(\lambda)$. Then $f(\lambda)$ is chosen as our composition; that is to say, the original $t_i \circ t_j$ is now replaced by $f(t_i \circ t_j)$. This means that $f(\lambda)$ are the elements of the set A ; and we have therefore to replace (3.1) and (3.2) by

$$\begin{aligned} [f(t_i \circ t_j)]^* &= f(t_i^* \circ t_j^*), & [f(t_i \circ t_j)]' &= f(t_i' \circ t_j') \\ \text{and} & & & \\ [f(\xi \circ \eta)]^* &= f(\xi^* \circ \eta^*), & [f(\xi \circ \eta)]' &= f(\xi' \circ \eta'). \end{aligned} \quad (4.2)$$

In this special case, suppose that the function satisfies the following distributive condition and that the composition is also associative:

$$\begin{aligned} f(f(t_i \circ t_j) \circ f(t_k \circ t_l)) &= ff(t_i \circ t_j \circ t_k \circ t_l), \\ f(f(t_i \circ t_j) \circ f(f(t_k \circ t_l) \circ f(t_m \circ t_n))) &= ff(t_i \circ t_j) \circ fff(t_k \circ t_l \circ t_m \circ t_n) \\ \dots & \dots \dots \dots \end{aligned}$$

Then, putting $ff = f^2$, $fff = f^3$, \dots , it is seen that for $p = 2^r$,

$$f^r(t_1 \circ t_2 \circ \dots \circ t_p) \quad (4.3)$$

is both self-associate and self-conjugate. And if the function is the identity, i.e., $f(t_i \circ t_j) = t_i \circ t_j$,

$$t_1 \circ t_2 \circ \dots \circ t_p \quad (4.4)$$

is both self-associate and self-conjugate.

Examples for the case $p = 4$. Let $a = l + im$, $i^2 = -1$, be a complex number $\neq 0$.

1. Let $a^* = -l + im$, $a' = l - im$. Then

$$\begin{aligned} t_1 &= a = a^{*/'} = a'^{*} = l + im, & t_2 &= a^* = a'^{*'} = -l + im \\ t_3 &= a^{*'} = a'^* = -l - im, & t_4 &= a^{*/} = a' = l - im \end{aligned}$$

The conditions (4.2) are here satisfied by the choice of addition as the composition, and any non-zero constant multiplier c as the function. Hence by (4.3)

$$c^2(t_1 + t_2 + t_3 + t_4) = 0$$

is the element which is both self-associate and self-conjugate.

2. Let $a^* = 1/(l + im)$, $a' = -(l + im)$. Then

$$t_1 = l + im, \quad t_2 = (l - im)/(l^2 + m^2), \quad t_3 = (-l + im)/(l^2 + m^2), \quad t_4 = -(l + im).$$

Proceeding as in the last example, we get the same result.

3. Let $a^* = 1/(l + im)$, $a' = l - im$. Then

$$t_1 = l + im, \quad t_2 = (l - im)/(l^2 + m^2), \quad t_3 = (l + im)/(l^2 + m^2), \quad t_4 = l - im.$$

The conditions (4.2) are here satisfied by the choice of multiplication as the composition and the identity as the function. Hence, by (4.4),

$$t_1 t_2 t_3 t_4 = 1$$

is the element which is both self-associate and self-conjugate.

4. Let $a^* = 1/(l-im)$, $a' = -l+im$. Then

$$t_1 = l+im, \quad t_2 = (l+im)/(l^2+m^2), \quad t_3 = (-l+im)/(l^2+m^2), \quad t_4 = -l+im.$$

The conditions (4.2) are here satisfied by the choice of multiplication as the composition and the positive square root as the function. Hence, by (4.1)

$$\sqrt{(\sqrt{(t_1 t_3)} \sqrt{(t_2 t_4)})} = i$$

is the element which is both self-associate and self-conjugate. Of course, all the four results are easily guessed from the operations of the associate and the conjugate.

5. Consider now the set A . Let the element $t_i \circ t_j$ of A be denoted by $t_{i,j}$. Since the composition is commutative, $t_{i,j} = t_{j,i}$. In what follows we shall use the abbreviations s.a. and s.c. to stand for the words 'self-associate' and 'self-conjugate', and shall suppose that $p > 2$ always.

Since the digits t_1, t_2, \dots, t_p form a cyclic sequence, we may regard $1, p$ as one of the pairs of consecutive numbers. If the indices i, j of $t_{i,j}$ are consecutive numbers, then $t_{i,j}$ is either s.a. or s.c. automatically; for example, $t_{1,2}^* = t_{1,2}$ identically. On the other hand, if there exists a s.a. element $t_{i,j}$ in which i, j are not consecutive numbers, then $t_{i,j}^* = t_{k,l}$, where i, j, k, l are distinct numbers. It will be advantageous to make a distinction between these two kinds of s.a. elements. We shall call the former kind, which is s.a. *digit by digit* (or, index by index), an *auto-s.a.* element. Similarly, for an *auto-s.c.* element. Such an element of either kind may be referred to as an *auto* element.

Let us start with an arbitrary element $t_{i,j}$ and proceed to construct the sequence of the type S_1 (See (2.1)). Three cases may arise: (1) $i = j$. (2) of the two indices i, j , one is odd and the other even, including the case when they are consecutive numbers and (3) $i \neq j$ are either both odd or both even.

(1) Here we obtain the following sequence of the type of a finite S_1 :

$$t_{1,1}, t_{2,2}, \dots, t_{p,p} \quad (5.1)$$

(2) Suppose we start with an auto-s.a. element $t_{r,r+1}$, where r is necessarily an odd number. Retaining only the terms which are not equal to a preceding term digit by digit, we get the following sequence:

$$t_{r,r+1}, t_{r-1,r+2}, \dots, t_{r+p/2+2, r+p/2-1}, t_{r+p/2+1, r+p/2} \quad (5.2)$$

The last term is the $(p/2)$ -th term in which the indices are consecutive numbers. And since $t_{r+s}^* = t_{r+s+1}$, if s is even, and $t_{r+s}^* = t_{r+s+1}$, if s is odd, the last term of (5.2) is an auto-s.a. or an auto-s.c. element according as $p/2$ is even or odd. Therefore, the sequence (5.2), although it is finite of order $\neq p/2$, is not cyclic unless $p = 4$. Similarly, if we start with an auto-s.c. element and proceed as before, we obtain a sequence of the type (5.2) with similar properties. It is also seen that if we start with any term of a sequence of the type (5.2), i.e., with an arbitrary element of A with one index odd and the other even, we get back the terms of the same sequence in which there are two auto terms, either of the same kind or of different kinds according as $p/2$ is even or odd, and the sequence so obtained is not cyclic unless $p = 4$.

(3) If we start with an element $t_{i,j}$, in which i, j are both odd or both even, the

resulting sequence will be such that the alternate terms have indices both odd and both even. So, there is no necessity to distinguish between the cases having indices both odd and both even. Suppose then we start with an element in which the indices are both odd and, without loss of generality, let this element be $t_{1,r}$, where r is odd. Then, remembering (2.2), we obtain the following sequence :

$$t_{1,r}, t_{2,r+1}, \dots, t_{p-r,p-1}, t_{p-r+1,p}, t_{p-r+2,1}, \dots, t_{p-1,p-2}, t_{p,r-1}. \quad (5.3)$$

This is a finite cyclic sequence of order $\nless p$. It is to be noticed that this sequence has the property that every index occurs twice, but not repeated in the same term, and that the difference between the values of the indices of a term is constant.

Let us now consider a special case. The case arises out of the possibility of two terms of (5.3), having a common index, being equal to one another, say the first term is equal to the $(p-r+2)$ -th term. Then

$$p-r+2 = r, \text{ or, } r = p/2+1. \quad (5.4)$$

Since r is odd, this case can occur if and only if $p/2$ is even. When this is so, the sequence (5.3) has its s -th term and $(p/2+s)$ -th term equal, $s = 1, 2, \dots$, and therefore the sequence reduces to

$$t_{1,p/2+1}, t_{2,p/2+2}, \dots, t_{p/2,p}. \quad (5.5)$$

This sequence is a finite cyclic sequence of order $\nless p/2$. Although (5.2) and (5.5) may have equal order, (5.5) has no auto term. Evidently, for a given value of p , there cannot exist more than one such sequence.

Now suppose, for the sake of convenience and for the time being, that the elements of the set A , which are not equal digit by digit, are distinct. Then the elements of A are $p(p+1)/2$ in number. Out of these, there are p elements which form the sequence (5.1). Out of the remaining $p(p-1)/2$ elements, there are p elements which are auto elements. They generate $p/2$ sequences of the type (5.2), each of order $p/2$. Thus there remains

$$R = p(p-1)/2 - p^2/4 = p(p-2)/4$$

elements. As $p > 2$ is even, $(p-2)/2$ is an integer.

(i) If $(p-2)/2 = 2r$ is even, $r > 0$, then $R = pr$. But then $p/2 = 2r+1$ is odd. Therefore in this case the R elements generate $r = (p-2)/4$ sequences of the type (5.3), each of order p .

(ii) If $(p-2)/2 = 2r+1$ is odd, $r \geq 0$, $R = pr + p/2$. But then $p/2 = 2(r+1)$ is even. Therefore in this case the R elements generate $r = (p-4)/4$ sequences of the type (5.3), each of order p and one sequence of the type (5.5) of order $p/2$. We thus have the following scheme :

| $p/2$ | No. of sequences of the type (5.1) | No. of sequences of the type (5.2) | No. of sequences of the type (5.3) | No. of sequences of the type (5.5) |
|-------|------------------------------------|------------------------------------|------------------------------------|------------------------------------|
| odd | 1 | $p/2$ | $(p-2)/4$ | nil |
| even | 1 | $p/2$ | $(p-4)/4$, if $p > 4$ | 1 |

Actually, however, the number of distinct elements of A may not be $p(p+1)/2$ for all values of p , as we shall see in the next article. We have accordingly to make necessary modifications.

6. Let us now examine whether any element $t_{i,j}$ of A can be both s.a. and s.c. Obviously, if $t_{i,j}$ is both s.a. and s.c., all the terms of the sequence generated by it have the same property and are equal to one another. It is also obvious that no two of the sequences considered in the last article can be equal.

Consider the sequence (5.1). It is seen intuitively that no term of this sequence can be both s.a. and s.c., because the term is concerned with one digit only and therefore takes no notice of the notions of associate and conjugate. It may however appear that the choice of division as the composition may make the terms equal, but then division is not commutative. Again, no term of the sequence of the type (5.3) can be both s.a. and s.c., unless $p = 4$, because we can always select, from such a sequence, a pair of terms having one common index while the other indices are distinct, e.g., $t_{i,j}$, $t_{i,k}$, $j \neq k$, and two such terms cannot be equal. In order therefore to look for an element of the set A which is both s.a. and s.c., we have to examine sequences of the types (5.2) and (5.5) when $p/2$ is even and of the type (5.2) only when $p/2$ is odd.

Consider an abstract element b and suppose that the sequence of the type S_1 generated by it is cyclic and of order p . If now we impose the further condition that b is s.a., then the first term of the sequence becomes equal to the second term, and the $(2+r)$ -th term = the $(p-r+1)$ -th term, $r = 1, 2, \dots, p/2-1$. Thus the number of distinct terms of the sequence is halved, and the sequence stands as

$$b = b^*, b', b'^*, \dots, b'^{*'} \dots$$

There are $p/2$ terms and the last term is s.a. or s.c. according as $p/2$ is even or odd. On the other hand, if we impose the condition that b is s.c., we obtain similarly the sequence

$$b = b', b^*, b'^*, \dots, b'^{*'} \dots$$

of $p/2$ terms, the last term being s.c. or s.a. according as $p/2$ is even or odd.

Analogously, take an element $t_{i,j}$ of A and impose the above conditions. Then, since t_i and t_j are each of order p , the corresponding sequences, each with $p/2$ terms, are

$$t_{i,j} = t_{i,j}^*, t_{i,j}', t_{i,j}^{**}, \dots, t_{i,j}'^{*'} \dots \quad (6.1)$$

$$t_{i,j} = t_{i,j}', t_{i,j}^*, t_{i,j}^{**}, \dots, t_{i,j}'^{*'} \dots \quad (6.2)$$

where the last terms have analogous properties. The terms of (6.1) or (6.2), however, are not necessarily distinct. If now $t_{i,j}$ happens to be both s.a. and s.c., the two sequences must be equal, except for sense. That is to say, for every term of one sequence, there is a term of the other sequence which is equal to it, digit by digit. Two cases arise: (1) i, j are both odd or both even, (2) of the indices i, j one is odd and the other even.

(1) Without loss of generality, we may suppose i, j both odd and, in particular, $i = 1$. Then (6.1) and (6.2) can be written as

$$t_{1,j}, t_{p,j-1}, t_{p-1,j-2}, \dots, t_{p/2+2,j+1-p/2}$$

$$t_{1,j}, t_{2,j+1}, t_{3,j+2}, \dots, t_{p/2,j-1+p/2}$$

Denote the terms of these two sequences by $l_1, l_2, \dots, l_{p/2}$ and $m_1, m_2, \dots, m_{p/2}$ respectively. If now we set

$$l_{1+r} = m_{1+p/2-r}, \quad r = 1, 2, \dots, p/2-1,$$

it is only then that the condition of equality of the two sequences is satisfied if and only if

$$j = p/2 + 1, \quad \text{and} \quad \therefore \quad p/2 \text{ is even}$$

Thus, the element $t_{1,p/2+1}$ is both s.a. and s.c. if and only if $p/2$ is even. In this case the two sequences reduce to the type (5.5).

(2) In this case, as we know, (6.1) and (6.2) contain terms which are auto elements, and therefore the method used in (1) cannot be applied here.

In the particular case when $p = 4$, we may however obtain information on reference to the examples given at the end of §4. It is seen that $t_{1,2}$ in Ex. 3, $t_{1,3}$ in Ex. 1, 4 and $t_{1,4}$ in Ex. 2 are elements which are both s.a. and s.c. But no two of these elements have this property in the same Ex., and this agrees with our assumption.

7. The properties of our algebraic system that we have obtained so far have an important application in Riemannian geometry which we give below. In what follows, the usual notations of tensor calculus are adopted.

Let there be a Riemannian space whose metric is given, as usual, by

$$ds^2 = g_{ij} dx^i dx^j,$$

and in the space let there be a law of parallel displacement of a contravariant vector defined, as usual, by

$$dV^i + \Gamma_{ij}^i V^j dx^j = 0,$$

where the coefficient of affine connection Γ_{ij}^i is supposed to be arbitrary. Denote the covariant derivative of a tensor with respect to this parallel displacement by a comma followed by an index. Put

$$a = \Gamma_{ij}^i, \quad a^* = \Gamma_{ij}^j + g^{ik} g_{ik,j}, \quad a' = \Gamma_{ji}^j. \quad (7.1)$$

The associate a^* and the conjugate a' of a (in accordance with the nomenclature adopted previously) in (7.1) are also affine connections. We may therefore form covariant derivatives of a tensor with respect to parallel displacements corresponding to a^* and a' . If a semi-colon followed by an index denote the covariant derivative of a tensor with respect to the parallel displacement corresponding to a^* , then it is known (Sen, 1948) that

$$g_{ik,j} + g_{ik,j} = 0.$$

It therefore follows that

$$a^{**} = a, \quad a'' = a.$$

Now construct the sequence S_1 as in (2.1):

$$a, \quad a^*, \quad a^{*'}, \quad a^{**}, \dots$$

The successive terms t_1, t_2, \dots of this sequence are all affine connections. The values of these terms can be obtained directly by calculating the covariant derivatives of the g_{ij} 's with respect to the different parallel displacements corresponding to the different terms t_i and also by interchanging the two appropriate lower indices in the covariant derivatives so obtained (Sen, 1949a). For example, it may be seen that

$$a^{*'} = a' + g^{ik}g_{jk,i}, \quad a^{**} = a^* + g^{ik}g_{is}(\Gamma_{kj}^s - \Gamma_{jk}^s) - g^{ik}g_{ij,k}.$$

If we put

$$\alpha = g^{ik}g_{ik,j}, \quad \alpha_o = g^{ik}g_{jk,i}$$

$$\beta = g^{ik}g_{is}(\Gamma_{kj}^s - \Gamma_{jk}^s), \quad \beta_o = g^{ik}g_{js}(\Gamma_{ki}^s - \Gamma_{ik}^s), \quad \gamma = g^{ik}g_{ij,k} = \gamma_o,$$

the terms of the above sequence have the following values which may be easily verified:

$$\begin{aligned} t_1 &= a &= a^{*/*/*/*/*/*} &= a \\ t_2 &= a^* &= a^{*/*/*/*/*/*} &= a + \alpha \\ t_3 &= a^{*'} &= a^{*/*/*/*/*/*} &= a' + \alpha_o \\ t_4 &= a^{**} &= a^{*/*/*/*/*/*} &= a + \alpha + \beta - \gamma \\ t_5 &= a^{*/*} &= a^{*/*/*/*/*/*} &= a' + \alpha_o + \beta_o - \gamma \\ t_6 &= a^{*/*/*} &= a^{*/*/*/*/*/*} &= a + \alpha + \alpha_o + \beta + \beta_o - \gamma \\ t_7 &= a^{*/*/*'} &= a^{*/*/*/*/*/*} &= a' + \alpha + \alpha_o + \beta + \beta_o - \gamma \\ t_8 &= a^{*/*/*/*} &= a^{*/*/*/*/*/*} &= a' + \alpha_o + \beta + \beta_o - \gamma \\ t_9 &= a^{*/*/*/*'} &= a^{*/*/*/*/*/*} &= a + \alpha + \beta + \beta_o - \gamma \\ t_{10} &= a^{*/*/*/*/*} &= a^{*/*} &= a' + \alpha_o + \beta_o \\ t_{11} &= a^{*/*/*/*/*'} &= a^{**} &= a + \alpha + \beta \\ t_{12} &= a^{*/*/*/*/*/*} &= a' &= a'. \end{aligned}$$

It is thus seen that the above sequence is a finite cyclic sequence of order $p = 12$. Accordingly, the terms can now be called "digits". Further let

$$a = \Gamma_{ij}^i, \quad b = L_{ij}^i$$

be two affine connections, and let the covariant derivatives of a tensor with respect to the parallel displacements corresponding to a , b and $(a+b)/2$ be denoted respectively by a comma, a solidus and an ordinary bracket followed by indices. Then (Sen, 1949b)

$$a^* = a + g^{ik}g_{ik,j}, \quad b^* = b + g^{ik}g_{ik,j}, \quad [\tfrac{1}{2}(a+b)]^* = \tfrac{1}{2}(a+b) + g^{ik}(g_{ik})_j.$$

Therefore

$$\begin{aligned} \tfrac{1}{2}(a^* + b^*) &= \tfrac{1}{2}(a+b) + \tfrac{1}{2}g^{ik}\{g_{ik,j} + g_{ik,j}\} \\ &= \tfrac{1}{2}(a+b) + \tfrac{1}{2}g^{ik}[2\frac{\partial g_{ik}}{\partial x^j} - g_{is}(\Gamma_{kj}^s + L_{kj}^s) - g_{ks}(\Gamma_{ij}^s + L_{ij}^s)] \\ &= \tfrac{1}{2}(a+b) + g^{ik}(g_{ik})_j = [\tfrac{1}{2}(a+b)]^* \end{aligned}$$

Thus we obtain the following property for every pair of digits, corresponding to (8.1):

$$[\tfrac{1}{2}(t_i + t_j)]^* = \tfrac{1}{2}(t_i^* + t_j^*), \quad [\tfrac{1}{2}(t_i + t_j)]' = \tfrac{1}{2}(t_i' + t_j') \quad (7.2)$$

We may now construct our finite set A with elements $t_{i,j} = t_i \circ t_j$ by choosing the composition as $t_i \circ t_j = \frac{1}{2}(t_i + t_j)$. We may then construct our algebraic system S as given in §3, all the conditions of the system being here satisfied. We may therefore apply the properties of the system in order to obtain the element of S which is both s.a. and s.c. In doing so we notice that $p/2$ is here even, and therefore the results of §6 are applicable. Hence, every term of the sequence generated by $t_{i,j}$ are equal elements having the desired property. From the table of values given above, we find that this element of S which is both s.a. and s.c. is

$$\frac{1}{2}(a + a^{**}) = \frac{1}{2}(a + a' + \alpha + \alpha_0 + \beta + \beta_0 - \gamma) \quad (7.3)$$

To understand the nature of the affine connection represented by (7.3), we have only to refer to (7.1) and notice that a s.a. parallel displacement is one with respect to which the covariant derivatives of the g_{ij} 's vanish and a s.c. parallel displacement is one with symmetric connection (i.e., where $\Gamma_{ij}^l = \Gamma_{ji}^l$). It is known (Eisenhart, 1927) that the only parallel displacement in Riemannian space which is both s.a. and s.c. is that of Levi-Civita. Therefore, the affine connection represented by (7.3) should be the Christoffel symbol $\left\{ \begin{smallmatrix} l \\ ij \end{smallmatrix} \right\}$. As a matter of fact, it is known (Sen, 1949a) that, in terms of symmetric connections,

$$\text{where} \quad \left\{ \begin{smallmatrix} l \\ ij \end{smallmatrix} \right\} = \frac{1}{2}(\Delta_{ij}^l - \nabla_{ij}^l) + \Omega_{ij}^l,$$

$$\nabla_{ij}^l = \frac{1}{2}(a + a') = b, \text{ say,}$$

$$\Delta_{ij}^l = \frac{1}{2}(b^* + b^{*'}) = c, \text{ say, } \Omega_{ij}^l = \frac{1}{2}(c^* + c^{*'}).$$

Therefore

$$\begin{aligned} \left\{ \begin{smallmatrix} l \\ ij \end{smallmatrix} \right\} &= \frac{1}{8}(a^* + a^{*'} + a^{**} + a^{*'*'} + a^{***} + a^{**'*'} + a^{****} + a^{***'*'} + a^{*****}) \\ &= \frac{1}{2}(a + a' + \alpha + \alpha_0 + \beta + \beta_0 - \gamma). \end{aligned}$$

Therefore, (7.3) is the Christoffel symbol, as is to be expected.

In accordance with the assumption, made in §3, that the system S cannot have more than one element which is both s.a. and s.c., the system possesses elements which are s.a. but not s.c., and also elements which are s.c. but not s.a. for certain values of p at least; e.g., all auto elements are such elements of one or the other kind in the case when $p/2 > 2$ is an even number. Naturally, this property has its application in Riemannian geometry; e.g., an affine connection which is s.a. but not s.c. is

$$\frac{1}{2}(a + a^*) = \Gamma_{ij}^j + \frac{1}{2}g^{ik}g_{ik}. \quad (7.4)$$

It is interesting to verify that the parallel displacement corresponding to (7.4) has the property that the covariant derivatives of the g_{ij} 's with respect to this parallelism vanish.

Let us now introduce an orthogonal ennuple ${}^T h_i$ at each point of the space defined, as usual, by

$$g_{ij} = \sum_T {}^T h_i {}^T h_j, \quad {}^T h^i = g^{ij} {}^T h_j.$$

It is then known (Einstein, 1928) that the parallel displacement corresponding to the affine connection

$${}^T h^i \frac{\partial {}^T h_i}{\partial x^j},$$

which is known as Einstein's teleparallelism, possesses the property that it is s.a. but not s.c. In order to find the nature of connection between these two parallelism, both of which are s.a. but not s.c., we notice that

$${}^T h^i \frac{\partial {}^T h_i}{\partial x^j} = \Gamma_{ij}^j + {}^T h^i \left(\frac{\partial {}^T h_i}{\partial x^j} - {}^T h_k \Gamma_{ij}^k \right) = \Gamma_{ij}^j + \sum_T g^{ik} {}^T h_k {}^T h_{i,j}.$$

Therefore, if the right hand side is to be equal to (7.4), we must have

$$\sum_T {}^T h_i {}^T h_{k,i} = \sum_T {}^T h_k {}^T h_{i,j}. \quad (7.5)$$

Hence, the parallelism corresponding to (7.4) reduces to Einstein's teleparallelism if and only if (7.5) is satisfied.

Thus, given an arbitrary affine connection $a = \Gamma_{ij}^j$ in a Riemannian space, it is possible to obtain both the Levi-Civita parallelism and Einstein's teleparallelism from the point of view of an algebraic system generated by a . Although Levi-Civita parallelism is the only one which is both s.a. and s.c., Einstein's teleparallelism is not the only one which is s.a. but not s.c..

In conclusion, it may be remarked that the basic ideas of the algebraic system given in this paper can be developed and further interesting properties can be obtained; and it is believed that a complete theory may be established.

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A NOTE ON BALANCED INCOMPLETE BLOCK DESIGNS

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1. Let v varieties be arranged in b blocks of k plots each, every variety being replicated r times and every pair of varieties occurring λ times. It is well known that the parameters v, b, r, k and λ satisfy the two relations:

$$vr = bk, \quad (1)$$

$$\lambda(v-1) = r(k-1). \quad (2)$$

Let l_{ij} ($i, j = 1, 2, \dots, b$) denote the number of varieties common between the i th and j th blocks. In the present note I give (i) a new proof of the inequality $b \geq v$, (ii) a simpler demonstration of a result of Schützenberger (1949) and (iii) the result that the value of the determinant $|l_{ij}| = 0$ if $b > v$ and $= r^2(r-\lambda)^{v-1}$ if $b = v$. The last result is believed to be new.

2. Let n_{ij} ($i = 1, 2, \dots, b; j = 1, 2, \dots, v$) denote 1 if the j th variety occurs in the i th block and 0 otherwise. Let β_j ($j = 1, 2, \dots, v$) denote the b -vector

$$(n_{1j}, n_{2j}, \dots, n_{bj}).$$

Evidently

$$\beta_i \cdot \beta_i = r \quad \text{and} \quad \beta_i \cdot \beta_j = \lambda \quad \text{if} \quad i \neq j.$$

The following system of b -vectors, v in number,

$$\alpha_1 = \frac{1}{\sqrt{r}} \beta_1$$

$$\alpha_j = \left\{ \beta_j - \frac{\lambda}{r + (j-2)\lambda} [\beta_1 + \beta_2 + \dots + \beta_{j-1}] \right\} / \sqrt{\left[r - \frac{\lambda^2(j-1)}{r + (j-2)\lambda} \right]}, \quad j = 2, 3, \dots, v$$

form an orthogonal system, each having unit length and each two being mutually orthogonal. As the maximum number of non-null orthogonal b -vectors is b , $v \leq b$.

Incidentally the vectors $\beta_1, \beta_2, \dots, \beta_v$ are independent and the rank of the matrix $\|n_{ij}\|$ is v .

3. Let $b = v$; then by (1), $r = k$. The square of the v -rowed determinant $|n_{ij}|$

$$|n_{ij}|^2 = \begin{vmatrix} r & \lambda & \lambda & \dots & \lambda \\ \lambda & r & \lambda & \dots & \lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda & \lambda & \lambda & \dots & r \end{vmatrix}.$$

Adding up the remaining rows to the first row, $r + \lambda(v-1) = r^2$ comes out as a common factor and the first row becomes $1, 1, \dots, 1$. Subtracting λ times the first row from the second, third, \dots , v th row, we have

$$|n_{ij}|^2 = r^2 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & r-\lambda & 0 & 0 \\ 0 & 0 & r-\lambda & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & r-\lambda \end{vmatrix} = r^2(r-\lambda)^{v-1}.$$

Therefore

$$|n_{ij}| = \pm r(r-\lambda)^{\frac{1}{2}(v-1)}.$$

The left hand side is an integer. Hence when v is even, $(r-\lambda)$ must be a perfect square. Thus a symmetrically balanced incomplete block design with an even number of varieties is impossible unless $(r-\lambda)$ is a perfect square. Recently Schützenberger (1949) obtained this result by a slightly more complicated method.

4. It is easy to verify

$$\sum_{j=1}^b l_{ij} = kr, \quad (3)$$

$$\sum_{i=1}^b l_{ij}^2 = k[r + (k-1)\lambda], \quad (i = 1, 2, \dots, b). \quad (4)$$

Let i be fixed; for the sake of simplicity we can take $i = 1$. Then

$$\sum_{j,j'=2}^b (l_{1j} - l_{1j'})^2 = 2(b-1) \sum_{j=2}^b l_{1j}^2 - 2 \left(\sum_{j=2}^b l_{1j} \right)^2 = 2k[(r-k)(b-1) + (k-1)\lambda(b-1) - k(r-1)^2].$$

When $r = k$, $v = b$, the right hand side = 0 and $l_{1j} = l_{1j'} = \frac{k(r-1)}{v-1} = \lambda$. A similar result follows for other values of i . Thus when $v = b$, $l_{ij} = \lambda$ if $i \neq j$.

5. Let $v = b$. We have $l_{ii} = k = r$ and $l_{ij} = \lambda$, $i \neq j$ and the determinant

$$|l_{ij}| = \begin{vmatrix} r & \lambda & \lambda & \dots & \lambda \\ \lambda & r & \lambda & \dots & \lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda & \lambda & \lambda & \dots & r \end{vmatrix} = r^2(r-\lambda)^{v-1}.$$

Let $b \geq v$. Then the square of the $(b+v)$ -rowed determinant

$$\begin{vmatrix}
n_{11} & n_{12} & \dots & n_{1v} & 0 & 0 & \dots & 0 \\
n_{21} & n_{22} & \dots & n_{2v} & 0 & 0 & \dots & 0 \\
. & . & . & . & . & . & . & . \\
n_{b1} & n_{b2} & \dots & n_{bv} & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & n_{11} & n_{21} & \dots & n_{b1} \\
0 & 0 & \dots & 0 & n_{12} & n_{22} & \dots & n_{b2} \\
. & . & . & . & . & . & . & . \\
0 & 0 & \dots & 0 & n_{1v} & n_{2v} & \dots & n_{bv}
\end{vmatrix} 2 = \begin{vmatrix}
l_{11} & l_{12} & \dots & l_{1b} & 0 & 0 & \dots & 0 \\
l_{21} & l_{22} & \dots & l_{2b} & 0 & 0 & \dots & 0 \\
. & . & . & . & . & . & . & . \\
l_{b1} & l_{b2} & \dots & l_{bb} & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & r & \lambda & \dots & \lambda \\
0 & 0 & \dots & 0 & \lambda & r & \dots & \lambda \\
. & . & . & . & . & . & . & . \\
0 & 0 & \dots & 0 & \lambda & \lambda & \dots & r
\end{vmatrix} = |l_{ij}| rk(r-\lambda)^{v-1}.$$

Since $b \neq v$, the $(b+v)$ -rowed determinant in the left hand side is zero as can be easily seen by developing the determinant by Laplace's method, picking out the first b rows and noting that either a b -rowed determinant that can be formed from these rows or its algebraic complement is zero. Hence when $b \neq v$, we get

$$\det |l_{ij}| = 0.$$

6. Let L_{ij} denote the cofactor of l_{ij} in the determinant $|l_{ij}|$. Then remembering (8) we easily get if $b \neq v$

$$\sum_i L_{ij} = \sum_j L_{ij} = 0, \quad i, j = 1, 2, \dots, b.$$

When $b = v$, the corresponding result is

$$\sum_i L_{ij} = \sum_j L_{ij} = (r-\lambda)^{v-1} \quad i, j = 1, 2, \dots, v.$$

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NOTE ON LAGUERRE'S POLYNOMIAL $L_n(z)$ AND ITS ASSOCIATED EQUATIONS (FUNCTIONAL AND DIFFERENTIAL)

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INTRODUCTION

The present investigation, as its title implies, relates principally to the subject of Laguerre's polynomial† $L_n(z)$, considered in relation to the two associated functional equations:

$$f'_n(z) = n[f'_{n-1}(z) - f_{n-1}(z)], \quad (\text{I})$$

and

$$f_{n+1}(z) - (2n+1-z)f_n(z) + n^2 f_{n-1}(z) = 0 \quad (\text{II})$$

and the differential equation of the second order, *vis.*,

$$z \frac{d^2 w}{dz^2} + (1-z) \frac{dw}{dz} + nw = 0, \quad (n = \text{an integer} \geq 0). \quad (\text{A})$$

For felicity of expression, (A) will be referred to as Laguerre's equation of *rank* n , and will be symbolised as $L^{(n)}$. The parameter n will be in most cases supposed to be an integer ≥ 0 , although there are casual references to *non-integral* parameters.

It is common knowledge that the polynomial $L_n(z)$, although introduced into analysis by Laguerre, has been commented upon by a band of prominent mathematicians, notably Tscheytscheff, Hilbert, Courant, Hahn, Szegő and Sonin. In the present set-up, the subject has been approached from the stand-point of Calculus of Functions or of Finite Differences.

As a matter of convenience, the paper has been divided into two sections. Section I deals firstly with certain characteristic properties of an *arbitrary* solution of (II) and secondly with a *common solution* $\{f_n(z)\}$ of the pair of simultaneous equations (I) and (II) and their indirect bearing on the differential equation (A). In the discussion of the sequence of *common* solutions (just mentioned), the two functions $f_{n-1}(z)$ and $f_{n+1}(z)$ separated by $f_n(z)$ have been said to be "contiguous" to $f_n(z)$. Finally, Section II treats of the *generating function* of the set of functions $\{f_n(z)/n!\}$, where $f_n(z)$ is an analytic

† Incidental references are made in the main body of the paper to the so-called "generalised" polynomial $\Pi_n(z)$, conceived of originally by Angelescu and modified subsequently by B. S. Sastry (1939). The function is known to satisfy (I) but not (II).

solution of (II). This is followed by a passing reference to the "generating function" of the sequence of functions $\{L_n(z)/n!\}$.

In conclusion we beg to express our gratefulness to our learned referee for his helpful criticisms and suggestive comments.

SECTION I

1. We know that the Laguerre's polynomial $L_n(z)$ satisfies each of the two functional equations:

$$f'_n(z) = n[f'_{n-1}(z) - f_{n-1}(z)], \quad (I)$$

and

$$f_{n+1}(z) - (2n+1-z)f_n(z) + n^2 f_{n-1}(z) = 0 \quad (II)$$

as well as the L -equation of rank $n(L^{(n)})$ viz.,

$$z \frac{d^2 w}{dz^2} + (1-z) \frac{dw}{dz} + nw = 0, \quad [w = f_n(z)] \quad (A)$$

whereas the "modified" Angelescu's polynomial* $\Pi_n(z)$, noted below, satisfies only (I) but not (II) or (A).

The main purpose of Section I is to reckon, in its most general aspect, with a class of enumerable functions $\{f_n(z)\}$, which shall satisfy the two simultaneous equations (I) and (II). Before we formally tackle this problem, it is worth while to establish a lemma, viz., that whenever a sequence of functions conforms to both (I) and (II), it must as a matter of course conform also to (A).

For, $(n+1)$ being written for n in (I), we have

$$f'_{n+1}(z) = (n+1)[f'_n(z) - f_n(z)]. \quad (1)$$

If (II) be differentiated and the derived relation be coupled with (1) so as to get rid of $f'_{n+1}(z)$, the resulting eliminant becomes

$$(n-z)f'_n(z) + nf_n(z) - n^2 f'_{n-1}(z) = 0. \quad (2)$$

The equation (I) being now solved as a *linear* differential equation in $f_{n-1}(z)$ gives

$$f_{n-1}(z) = \frac{f_n(z)}{n} + \frac{ae^z}{n} + \frac{1}{n} \cdot e^z \int e^{-z} f_n(z) dz, \quad (3)$$

where a is a constant.

If the value of $f'_{n-1}(z)$, derived from (3) by differentiation, be inserted in (2), we readily find

* Angelescu's function $\Pi_n(z)$, as modified by B. S. Shastri (1939), is defined by :

$$\Pi_n(z) = e^z \left(\frac{d}{dz} \right)^n [e^{-z} A_n(z)],$$

where $A_n(z)$ stands for the polynomial $(a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n)$.

$$zf'_n(z) + nae^z + ne^z \int e^{-z} f_n(z) dz = 0. \quad (4)$$

Eliminating the set of terms

$$nae^z + ne^z \int e^{-z} f_n(z) dz$$

from the equation (4) and the second equation derived from it by simple differentiation, we are led to the relation

$$zf'_n(z) + (1-z)f'_n(z) + nf_n(z) = 0,$$

showing that $f_n(z)$ is a solution of (A). This proves the premised lemma, which may also be worded as follows:

The sequence of functions $\{f_n(z)\}$, satisfying the two simultaneous functional equations (I) and (II), must ipso facto satisfy the sequence of differential equations $\{L^{(n)}\}$.

2. In this article we propose to devise a synthetic construction of the *common* solutions of the two functional equations (I) and (II). The key-note to this problem is contained implicitly in Art. 1, where it is proved formally that *every* function $f_n(z)$, satisfying both (I) and (II), must satisfy the associated equation $L^{(n)}$.

The precise process to be adopted is to start with an arbitrarily assigned solution $\{f_n(z)\}$ of the differential equation $L^{(n)}$ of a *prescribed* rank $n (\geq 1)$, so that

$$zf'_n(z) + (1-z)f'_n(z) + nf_n(z) = 0. \quad (1)$$

A first integral of this is

$$zf'_n(z) = -ne^z \left[\int e^{-z} f_n(z) dz + c \right], \quad (2)$$

where the constant c is to be treated as *known* on the ground that the particular solution $f_n(z)$ itself is *known*.

If we now introduce the two "contiguous" functions $f_{n-1}(z)$ and $f_{n+1}(z)$ in accordance with the pair of relations

$$f'_n(z) = n[f'_{n-1}(z) - f_{n-1}(z)] \quad (I)$$

and

$$f_{n+1}(z) - (2n+1-z)f_n(z) + n^2 f_{n-1}(z) = 0, \quad (II)$$

we have on integrating (I), as in Art. 1, for $f_{n-1}(z)$,

$$f_{n-1}(z) = \frac{1}{n} \left[f_n(z) + e^z \int e^{-z} f_n(z) dz + ae^z \right], \quad (3)$$

where a is a constant, as yet undetermined.

Admittedly the *contiguous* function $f_{n-1}(z)$, viz. (3), containing as it does the *arbitrary* element a , is not exactly determinate. But this *arbitrariness* disappears as soon as $f_{n-1}(z)$ is restricted to satisfy the equation $L^{(n-1)}$.

For, if the values of $f'_{n-1}(z)$ and $f''_{n-1}(z)$, as obtained from (3) by two-fold differentiation be made use of, the left side of the differential equation $L^{(n-1)}$, viz.,

$$zf''_{n-1}(z) + (1-z)f'_{n-1}(z) + (n-1)f_{n-1}(z)$$

simplifies, by virtue of (1), to the form

$$\frac{1}{n} \left[zf'_n(z) + n \left(e^z \int e^{-z} f_n(z) dz + a e^z \right) \right],$$

which further simplifies to

$$(a-c)e^z$$

in view of (2).

It follows, then, that $f_{n-1}(z)$ will satisfy $L^{(n-1)}$, if and only if

$$a-c=0.$$

Thus the constant a , occurring in (3), which was up till now undefined, becomes now perfectly determinate and is in fact equal to the *known* constant c , occurring initially in (2). So (3) may now be presented in the form

$$f_{n-1}(z) = \frac{1}{n} \left[f_n(z) + e^z \int e^{-z} f_n(z) dz + c e^z \right]. \quad (4)$$

Elimination of the set of terms

$$e^z \int e^{-z} f_n(z) dz + c e^z$$

from (2) and (4) leads to the *subsidiary* relation

$$zf'_n(z) - n f_n(z) + n^2 f_{n-1}(z) = 0, \quad (5)$$

which will be utilised elsewhere.

If we now take stock of what has been done so far, we observe that, for an *assigned* particular solution $f_n(z)$ of the differential equation $L^{(n)}$ of *assigned* rank, the equation (4) serves, to determine the "contiguous" function $f_{n-1}(z)$ *uniquely*. That being so, the other functional equation (II) serves to determine the second "contiguous" function $f_{n+1}(z)$ with similar *uniqueness*. It is now a pleasant job to verify that the function $f_{n+1}(z)$, thus defined, automatically satisfies the equation $L^{(n+1)}$.

For (II) being re-written as

$$f_{n+1}(z) = (2n+1-z)f_n(z) - n^2 f_{n-1}(z), \quad (6)$$

we have

$$f'_{n+1}(z) = (2n+1-z)f'_n(z) - f_n(z) - n^2 f'_{n-1}(z). \quad (7)$$

Eliminating $f_{n-1}(z)$ and $f'_{n-1}(z)$ linearly from (I), (5) and (7), we deduce after easy reductions

$$f'_{n+1}(z) = (n+1)\{f'_n(z) - f_n(z)\}. \quad (8)$$

Computing the value of $f''_{n+1}(z)$ and substituting, we have, by (1) and (5)

$$\begin{aligned}
& z f_{n+1}''(z) + (1-z) f_{n+1}'(z) + (n+1) f_{n+1}(z) \\
&= (n+1) [z \{f_n''(z) - f_n'(z)\} + (1-z) \{f_n'(z) - f_n(z)\} + (2n+1-z) f_n'(z) - n^2 f_{n-1}(z)] \\
&= (n+1) [z f_n''(z) + (1-z) f_n'(z) + 2n f_n(z) - z f_n'(z) - n^2 f_{n-1}(z)] \\
&= (n+1) [n f_n(z) - z f_n'(z) - n^2 f_{n-1}(z)], \\
&= 0.
\end{aligned}$$

Thus $f_{n+1}(z)$ satisfies the differential equation $L^{(n+1)}$.

Inasmuch as (8) is almost the same as (I) and in fact differs from it only in having $(n+1)$ in place of n , it is plain that $f_n(z)$ is one the two functions "contiguous" to $f_{n+1}(z)$. If, then, the second "contiguous" function $f_{n+2}(z)$ of $f_{n+1}(z)$ be defined by

$$f_{n+2}(z) - (2n+3-z) f_{n+1}(z) + (n+1)^2 f_n(z) = 0, \quad (9)$$

we can apply the same mode of reasoning to prove that $f_{n+2}(z)$ satisfies $L^{(n+2)}$. Repetition of the same line of argument leads ultimately to an "ascending" sequence of functions, beginning with $f_{n+1}(z)$, *vis.*,

$$f_{n+1}(z), f_{n+2}(z), f_{n+3}(z), \dots, \quad (10)$$

which satisfy respectively the sequence of differential equations

$$L^{(n+1)}, L^{(n+2)}, L^{(n+3)}, \dots,$$

and between any two or any three consecutive members of which there shall subsist the two functional equations (I) and (II).

Proceeding along the same line *in the reverse order*, we shall similarly come across a "descending" sequence of functions, beginning with $f_{n-1}(z)$, *vis.*,

$$f_{n-1}(z), f_{n-2}(z), f_{n-3}(z), \dots, \quad (11)$$

which shall satisfy the respective differential equations of the sequence

$$L^{(n-1)}, L^{(n-2)}, L^{(n-3)}, \dots, L^{(0)},$$

and any two or any three consecutive members of which shall abide by the two equations (I) and (II).

Putting this and that together, we readily realise that a knowledge of a single particular solution $f_m(z)$ of a particular differential equation of assigned rank ($L^{(m)}$) serves in its own way to determine a unique set of functions like $\{f_n(z)\}$, which shall satisfy the two simultaneous equations (I) and (II) and, naturally therefore, the differential equations of the set $\{L^{(n)}\}$. Palpably the totality of common solutions $\{f_n(z)\}$ of the pair of equations (I) and (II) must be conventionally taken as ∞^2 , seeing that the initial choice of particular integral $f_n(z)$ of $L^{(n)}$ must needs involve two arbitrary constants.

Having thus disposed of a synthetic method of constructing the common solutions of (I) and (II), we shall devote Section II of the paper to the determination of the generating function of the sequence of functions $\{f_n(z)/n!\}$ where $\{f_n(z)\}$ is an arbitrary solution of (II).

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SECTION II

GENERATING FUNCTION OF THE SEQUENCE OF FUNCTIONS $\{f_n(z)/n!\}$,
WHERE $f_n(z)$ IS AN ARBITRARY SOLUTION OF (II)

3. Supposing $\{f_n(z)\}$ to be an arbitrary solution of the functional equation

$$f_{n+1}(z) - (2n+1-z)f_n(z) + n^2 f_{n-1}(z) = 0, \quad (\text{II})$$

and setting

$$\varphi_n(z) = f_n(z)/n!,$$

we easily deduce that the equation for $\varphi_n(z)$ is

$$(n+1)\varphi_{n+1}(z) - (2n+1-z)\varphi_n(z) + n\varphi_{n-1}(z) = 0. \quad (1)$$

If we now write

$$V = \sum_{n=0}^{\infty} h^n \varphi_n(z), \quad (2)$$

and assume for the present the validity of termwise differentiation (*w.r.t.* h) of the infinite series on the right side of (1) and attend to the relation (1), we find without much difficulty

$$(1-h)^2 \frac{\partial V}{\partial h} + (h-A)V = B, \quad (3)$$

where A and B are certain functions of z , defined by

$$\left. \begin{aligned} A &\equiv 1-z \\ B &\equiv \varphi_1(z) - (1-z)\varphi_0(z) = \varphi_1(z) - A\varphi_0(z) \end{aligned} \right\} \quad (4)$$

Solving (3) as a *linear* differential equation in V , we obtain

$$V = \frac{1}{(1-h)e^{\frac{z}{1-h}}} \left[B \int (1-h)e^{\frac{z}{1-h}} dh + \psi(z) \right] \quad (5)$$

where the constant of integration, *viz.*, $\psi(z)$ is independent of h and is certainly determinate as soon as a particular value is ascribed to h .

Reserving for the next article a discussion of the limitation to which the equality (5) is subject, let us consider the particular case where $f_n(z) \equiv L_n(z)$ for *every* n . This can be easily realised if we set

$$f_0'(z) = L_0(z) = 1 \quad \text{and} \quad f_1(z) = L_1(z) = 1-z, \quad (6)$$

for (II) is a linear difference equation of the *second* order.

Combining (4) and (6), we get

$$B = 0,$$

so that (5) simplifies to

$$\sum_{n=0}^{\infty} \frac{h^n L_n(z)}{n!} = \frac{\psi(z)}{(1-h)e^{z/(1-h)}}. \quad (7)$$

To find $\psi(z)$, we write $h = 0$ in (7), so as to deduce

$$L_0(z) = \psi(z)/e^z, \text{ i. e., } \psi(z) = e^z.$$

Accordingly (7) reduces to the *known* result†

$$\sum_{n=0}^{\infty} \frac{h^n L_n(z)}{n!} = \frac{e^{-zh/(1-h)}}{1-h}. \quad (8)$$

4. In order to investigate the condition under which the formula (5) of the previous article holds good, we may write

$$a_n \equiv \varphi_n(z)$$

in (1) and (2) of the same article. Thus we have

$$V = \sum_{n=0}^{\infty} a_n h^n, \quad (1)$$

where $\{a_n\}$ conforms to the recurrent relation

$$a_{n+2} = p_n a_{n+1} + q_n a_n, \quad (2)$$

it being understood that

$$p_n \equiv \frac{2n+3-z}{n+2} \quad \text{and} \quad q_n = -\frac{n+1}{n+2}.$$

In view of the equalities

$$\lim_{n \rightarrow \infty} p_n = 2 \quad \text{and} \quad \lim_{n \rightarrow \infty} q_n = -1,$$

we learn, on reckoning with the relation (2) and appealing to Van Vleck's (1900) famous theorem, that the radius of convergence (R) of the power-series in h , viz., (1), is equal to the smaller of the moduli of the roots of the quadratic in t , viz.,

$$t^2 - 2t + 1 = 0. \quad (3)$$

Both the roots of (3) being $= 1$, R must also be $= 1$. This radius of convergence (viz., unity) being independent of z , it follows that the equality (5) of Art. 3 is valid for *all* positions of h within the circle $|h| = 1$ on the h -plane and for *all* positions of z in the *finite* part of the z -plane, provided that ' z ' never coincides with a singularity of any function of the set $\{\varphi_n(z)\}$, i. e., of $\{f_n(z)\}$. Subject to these limitations on h and z , the

† In the above context the Laguerre polynomial $L_n(z)$ has been defined as that particular solution of the functional equation

$$f_{n+1}(z) - (2n+1-z)f_n(z) + n^2 f_{n-1}(z) = 0,$$

which fulfils the two special conditions

$$f_0(z) = 1 \quad \text{and} \quad f_1(z) = 1-z.$$

As shewn above this definition leads to the other definition (implied in (8), viz., that $L_n(z)$ is the coefficient of $h^n/n!$ in the expansion of $e^{-zh/(1-h)}/(1-h)$. Although this result is mentioned in Hobson's "*Spherical and Ellipsoidal Harmonics*" and proved differently by B. S. Sastri (1939), still it is inserted in the above set up inasmuch as it provides a simple illustration of the *general* summation-formula (5).

series on the L.S. of the formula (5) of Art. 3 is absolutely and uniformly convergent *w.r.t.* h . This justifies the operation of termwise differentiation (*w.r.t.* h), effected already in Art. 3 on the said formula.

If we now introduce the restriction that the functions of either set $\{f_n(z)\}$ or $\{\varphi_n(z)\}$, satisfying (II) of Art. 3, shall be *integral* (rational or transcendental), the possible existence of a singularity of any of the functions in the *finite* part of the z -plane has to be discounted and the formula in question holds for *all* finite positions of z , provided, of course, that $|h| < 1$.

The main set of results may then be finalised as follows:—

If $\{f_n(z)\}$ be an enumerable set of integral functions, satisfying the functional equation

$$f_{n+1}(z) - (2n+1-z)f_n(z) + n^2 f_{n-1}(z) = 0, \quad (n \geq 1), \quad (\text{II})$$

then the summation-formula

$$\sum_{n=0}^{\infty} \frac{h^n f_n(z)}{n!} = \frac{\{f_1(z) - (1-z)f_0(z)\} \cdot \left\{ \int (1-h)e^{z/(1-h)} dh + \psi(z) \right\}}{(1-h)e^{z/(1-h)}} \quad (4)$$

holds for all positions of z in the finite part of the z -plane and for all positions of ' h ' within the circle $|h| = 1$ in the h -plane, it being tacitly understood that the function $\psi(z)$, occurring in the R.S. of (4) is to be found by assigning a convenient admissible value to h .

The particular case when the integral function $f_n(z)$ happens to be rational and to coincide, in fact, with $L_n(z)$, has been noticed heretofore.

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ON GENERALISED LEGENDRE POLYNOMIALS

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1. Let D_k^k stand for the operator $\frac{d}{dz} \frac{1}{z^{k-1}} \frac{d}{dz}$ and D_k^{km} for the operator D_k^k repeated m times. Sharma (1948) has shown that

$${}_2F_1\left(-m, m + \frac{1}{k}; 1; 1 - z^k\right) = Q_{km}(z) = \frac{1}{k^{2m}(2m)!} D_k^{km}(1 - z^k)^{2m}.$$

When $k = 2$, this reduces to

$$P_{2m}(z) = \frac{1}{2^{2m}(2m)!} \frac{d^{2m}}{dz^{2m}} (1 - z^2)^{2m}$$

which is the well-known Rodrigue's formula (Whittaker and Watson, 1920) for a Legendre Polynomial. Let $A = 1/k^{2m}(2m)!$. Then

$$\begin{aligned} I &= \int_0^1 y^{kr} Q_{km}(y) dy = A \int_0^1 y^{kr} \left\{ \frac{d}{dy} \frac{1}{y^{k-1}} \frac{d}{dy} \right\} [D_k^{km-k}(1 - y^k)^{2m}] dy \\ &= -A \int_0^1 k r y^{kr-k+1} \frac{d}{dy} [D_k^{km-k}(1 - y^k)^{2m}] dy = A k r (kr - k + 1) \int_0^1 y^{kr-k} D_k^{km-k}(1 - y^k)^{2m} dy \\ &= A k (kr - k)(kr - k + 1)(kr - 2k + 1) \int_0^1 y^{kr-2k} D_k^{km-2k}(1 - y^k)^{2m} dy. \end{aligned}$$

Repeating this process of integration m times, we have the integral

$$\begin{aligned} I &= A k r (kr - k)(kr - 2k) \dots (kr - m - 1k)(kr - k + 1)(kr - 2k + 1) \dots (kr - mk + 1) \\ &\quad \times \int_0^1 y^{kr-km}(1 - y^k)^{2m} dy \\ &= \frac{A k^{2m-1} \Gamma(r+1/k) \Gamma(r+1) \Gamma(2m+1)}{\Gamma(r+m+1/k+1) \Gamma(r-m+1)} = \frac{1}{k} \frac{\Gamma(r+1/k) \Gamma(r+1)}{\Gamma(r+m+1/k+1) \Gamma(r-m+1)}; \quad (r \geq m) \\ &= 0, \quad (r < m). \end{aligned}$$

Again we note that

$$Q_{km+1}(z) = \frac{1}{k^{2m+1}(2m+1)!} \frac{d}{dz} D_k^{km}(1 - z^k)^{2m+1} = {}_2F_1(-m, m+1/k+1, 1; 1 - z^k).$$

* Editorial board regrets to report the untimely death of the author after the submission of this paper.

In a similar manner, integrating term by term, we get

$$\int_0^1 y^{kr} Q_{km+1}(y) dy = \frac{\Gamma(-1/k)}{\Gamma(m+1)\Gamma(-m-1/k)} \cdot \frac{1}{kr+2} \times {}_3F_2 \left(\begin{matrix} -m, m+1/k+1, r+2/k; \\ 1/k+1, r+2/k+1; \end{matrix} 1 \right).$$

The hypergeometric function is Saalschützian. Hence

$$\int_0^1 y^{kr} Q_{km+1}(y) dy = \frac{1}{k} \frac{\Gamma(r+1/k)\Gamma(r+2/k)}{\Gamma(r+1/k-m)\Gamma(r+m+2/k+1)}.$$

Let us next consider the integral

$$\begin{aligned} & \int_0^1 (1+y^k)^p Q_{km}(y) dy \\ &= \int_0^1 Q_{km}(y) \left\{ \frac{p(p-1) \dots (p-m+1)}{m!} y^{mk} + \frac{p(p-1) \dots (p-m)}{(m+1)!} y^{(m+1)k} + \dots \right\} dy \\ &= \frac{p(p-1) \dots (p-m+1)\Gamma(m+1/k)}{k\Gamma(2m+1/k+1)} \left\{ 1 + \frac{(p-m)(m+1/k)}{1!(2m+1/k+1)} \right. \\ & \quad \left. + \frac{(p-m)(p-m-1)(m+1/k)(m+1/k+1)}{2!(2m+1/k+1)(2m+1/k+2)} + \dots \right\} \\ &= \frac{1}{k} \frac{\Gamma(p+1)\Gamma(m+1/k)}{\Gamma(p-m+1)\Gamma(2m+1/k+1)} \cdot {}_2F_1(m-p, m+1/k; 2m+1/k+1; -1). \end{aligned}$$

As special cases, we get

$$\int_0^1 (1+y^k)^m Q_{km}(y) dy = \frac{1}{k} \frac{\Gamma(m+1)\Gamma(m+1/k)}{\Gamma(2m+1/k+1)}.$$

Since

$${}_2F_1(a, b; 1+a-b; -1) = \frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+\frac{1}{2}a-b)\Gamma(1+a)},$$

we get when $p = 2m$,

$$\int_0^1 (1+y^k)^{2m} Q_{km}(y) dy = \frac{1}{2k} \frac{\Gamma(2m+1)\Gamma(\frac{1}{2}m+1/2k)}{\Gamma(m+1)\Gamma(\frac{3}{2}m+1/2k+1)}.$$

The following integrals can be obtained in a similar manner.

$$\begin{aligned} & \int_0^1 (1-y^k)^p Q_{km}(y) dy = \frac{(-1)^m \{\Gamma(p+1)\}^2 \Gamma(m+1/k)}{k\Gamma(p-m+1)\Gamma(m+1)\Gamma(m+p+1/k+1)}, \\ & \int_0^1 (1+y^k)^p Q_{km+1}(y) dy = \frac{1}{k} \frac{\Gamma(1/k)\Gamma(2/k)}{\Gamma(1/k-m)\Gamma(m+2/k+1)} \\ & \quad \times {}_3F_2\{-p, 1/k, 2/k; 1/k-m, m+2/k+1; -1\}, \end{aligned}$$

$$\begin{aligned}\int_0^1 (1-y^k)^p Q_{km+1}(y) dy &= \frac{1}{k} \frac{\Gamma(1/k)\Gamma(2/k)}{\Gamma(1/k-m)\Gamma(m+2/k+1)} \\ &\quad \times {}_3F_2\{-p, 1/k, 2/k; 1/k-m, m+2/k+1; 1\}, \\ \int_0^1 (1+y^k)^p y^{k-1} Q_{km+1}(y) dy &= \frac{1}{k} \frac{\Gamma(p+1)\Gamma(m+1/k+1)}{\Gamma(p-m+1)\Gamma(2m+1/k+2)} \\ &\quad \times {}_2F_1(m-p, m+1/k+1, 2m+1/k+2; -1), \\ \int_0^1 (1-y^k)^p y^{k-1} Q_{km+1}(y) dy &= \frac{(-1)^m \{\Gamma(p+1)\}^2 \Gamma(m+1/k+1)}{k \Gamma(p-m+1) \Gamma(m+p+1/k+2) \Gamma(m+1)}.\end{aligned}$$

Special cases of the above are

$$\begin{aligned}\int_0^1 (1-y^k)^{m-1/k} Q_{km+1}(y) dy &= \frac{1}{k} \frac{\Gamma(1/k)\Gamma(2/k)\Gamma(m-1/k+1)}{\Gamma(m+1)\Gamma(1/k-m)\Gamma(m+1/k+1)}, \\ \int_0^1 (1+y^k)^m y^{k-1} Q_{km+1}(y) dy &= \frac{1}{k} \frac{\Gamma(m+1)\Gamma(m+1/k+1)}{\Gamma(2m+1/k+2)}, \\ \int_0^1 (1+y^k)^{2m} y^{k-1} Q_{km+1}(y) dy &= \frac{\Gamma(2m+1)\Gamma(\frac{3}{2}+\frac{1}{2}m+1/2k)}{\Gamma(m+1)\Gamma(\frac{3}{2}+\frac{3}{2}m+1/2k)(mk+k+1)}.\end{aligned}$$

2. Let us next evaluate the integral

$$\int_0^1 e^{-xy} Q_{km}(y) dy,$$

which is equal to

$$\int_0^1 \left\{1 - \frac{xy^k}{1!} + \frac{x^2 y^{2k}}{2!} - \dots\right\} Q_{km}(y) dy = \frac{(-1)^m}{k} \frac{\Gamma(m+1/k)}{\Gamma(2m+1/k+1)} x^{-(1+1/2k)} e^{-\frac{1}{2}x} M_{1/2k-\frac{1}{2}, m+1/2k}(x).$$

Let us multiply both sides by $e^{-ax} x^r$ and integrate with respect to x between the limits zero and infinity. Since

$$\int_0^\infty e^{-x(a+y^k)} x^r dx = \frac{\Gamma(r+1)}{(a+y^k)^{r+1}}.$$

Hence

$$\int_0^1 \frac{Q_{km}(y) dy}{(a+y^k)^{r+1}} = (-1)^m \frac{\Gamma(m+1/k)\Gamma(m+r+1)}{k \Gamma(r+1) \Gamma(2m+1/k+1) (1+a)^{m+r+1}}.$$

In a like manner we get

$$\begin{aligned}\int_0^1 e^{-xy} y^{kp} Q_{km+1}(y) dy &= \frac{1}{k} \frac{\Gamma(p+1/k)\Gamma(p+2/k)}{\Gamma(p-m+1/k)\Gamma(p+m+2/k+1)} \\ &\quad \times {}_2F_2(p+1/k, p+2/k; p-m+1/k, p+m+2/k+1; -x) \\ \int_0^1 \frac{y^{kp} Q_{km+1}(y) dy}{(a+y^k)^r} &= \frac{1}{k} \frac{\Gamma(p+1/k)\Gamma(p+2/k)}{\Gamma(p-m+1/k)\Gamma(p+m+2/k+1) a^{r+1}} \\ &\quad \times {}_3F_2(p+1/k, p+2/k, r+1; p-m+1/k, p+m+2/k+1; -1/a).\end{aligned}$$

The following are the special cases

$$\int_0^1 \frac{Q_{km}(y)dy}{(1+y^k)^{2m+2/k}} = \frac{1}{2} \frac{(-1)^m \Gamma(m+1/k) \Gamma(\frac{3}{2}m+1/k)}{k \Gamma(\frac{1}{2}m+1) \Gamma(2m+2/k)},$$

$$\int_0^1 \frac{Q_{km}(y)dy}{(1+y^k)^{m+1/k+1}} = \frac{(-1)^m}{(mk+1)2^{m+1/k}},$$

$$\int_0^1 \frac{y^{(m+1)k} Q_{km+1}(y)dy}{(1+y^k)^{2m+2/k+2}} = \frac{\pi^{\frac{1}{2}}}{2k} \frac{\Gamma(\frac{1}{2}m+1/k+\frac{1}{2})}{2^{2m+2/k+1} \Gamma(\frac{1}{2}-\frac{1}{2}m) \Gamma(m+\frac{1}{2}+1/k)}$$

showing that the integral vanishes when m is an odd positive integer.

3. We can also write

$$Q_{km}(y) = \frac{(-1)^m \Gamma(m+1/k)}{\Gamma(1/k) \Gamma(m+1)} {}_2F_1(-m, m+1/k; 1/k; y^k).$$

Hence

$$\begin{aligned} \int_0^1 y^{kr} Q_{km}(y) Q_{kn}(y) dy \\ = (-1)^m \frac{\Gamma(m+1/k)}{\Gamma(1/k) \Gamma(m+1)} \int_0^1 \left\{ y^{kr} + \frac{(-m)(m+1/k)}{1 \cdot 1(1/k)} y^{kr+k} + \dots \right\} Q_{kn}(y) dy \end{aligned}$$

Integrating term by term we get the value of the integral

$$\begin{aligned} = (-1)^m \frac{\Gamma(m+1/k) \Gamma(r+1) \Gamma(r+1/k)}{k \Gamma(1/k) \Gamma(m+1) \Gamma(n+r+1/k+1) \Gamma(-n+r+1)} \\ \times {}_4F_3(-m, r+1/k, r+1, m+1/k; n+r+1/k+1, 1/k, -n+r+1; 1). \end{aligned}$$

The hypergeometric series is Saalschützian. In a similar manner

$$\begin{aligned} \int_0^1 y^{kr} Q_{km+1}(y) Q_{kn+1}(y) dy = \frac{(-1)^n \Gamma(r+2/k) \Gamma(r+3/k) \Gamma(n+1/k+1) \Gamma(m+r+3/k+1)}{k \Gamma(n+1) \Gamma(1/k+1) \Gamma(-m+r+2/k)} \\ \times {}_4F_3\left(\begin{matrix} -n, n+1/k+1, r+2/k, r+3/k; \\ 1/k+1, -m+r+2/k, m+r+3/k+1; \end{matrix} 1\right) \end{aligned}$$

A special case of the above is

$$\begin{aligned} \int_0^1 y^{km+kn-k+1} Q_{km}(y) Q_{kn}(y) dy \\ = \frac{\Gamma(m+n+1) \{\Gamma(m+n+1/k)\}^2 \Gamma(m+n+2/k-1)}{k \Gamma(m+1) \Gamma(n+1) \Gamma(m+1/k) \Gamma(n+1/k) \Gamma(2m+2n+2/k)}. \end{aligned}$$

4. By Laplace's transform, we have, if

$$f(p) = p \int_0^\infty e^{-px} h(x) dx,$$

then

$$f(p) \doteq h(x) \quad \text{and} \quad 1/p^r \doteq x^r / \Gamma(r+1):$$

We have proved that

$$\int_0^1 e^{-xy} x^r Q_{km}(y) dy = \frac{(-1)^m \Gamma(m+1/k)}{k \Gamma(2m+1/k+1)} x^{m+r} {}_1F_1(m+1/k; 2m+1/k+1; -x).$$

Let us put $x = 1/p$. On interpretation and writing t^2 for t and putting $r = 1/k - 1$, we get

$$\int_0^1 y^{\frac{1}{2}(k-1)} J_{1/k-1}(2y^{\frac{1}{2}kt}) Q_{km}(y) dy = \frac{(-1)^m}{kt} J_{2m+1/k}(2t).$$

The following are the special cases

$$k = 1; \int_0^1 J_0(2y^{\frac{1}{2}t}) Q_m(y) dy = \frac{(-1)^m}{t} J_{2m+1}(2t);$$

$$k = 2; \int_0^1 P_{2m}(y) \cos 2ty dy = \frac{1}{2} (-1)^m (\pi/t)^{\frac{1}{2}} J_{2m+\frac{1}{2}}(2t).$$

In a similar manner we notice that

$$\begin{aligned} \int_0^1 y^{\frac{1}{2}kr+k-1} J_r(2y^{\frac{1}{2}kt}) Q_{km+1}(y) dy \\ = \frac{\Gamma(r+1/k+1)}{k \Gamma(-m+r+1) \Gamma(m+r+2+1/k)} t^{r-1} {}_1F_2(r+1/k+1, -m+r+1; m+r+1/k+2; -t^2) \end{aligned}$$

$$\begin{aligned} \int_0^1 y^{\frac{1}{2}kr+k-2} J_r(2y^{\frac{1}{2}kt}) Q_{km+1}(y) dy \\ = \frac{\Gamma(r-1/k+1)}{k \Gamma(-m+r-1/k+1) \Gamma(m+r+2)} {}_1F_2(r-1/k+1; -m+r-1/k+1, m+r+2; -t^2). \end{aligned}$$

5. We shall now find some integral and other representations of $Q_{km}(z)$.

We shall prove that

$$\int_0^\infty e^{-(1-s^2)t^2} D_k^{kn} (e^{-s^2 t^2}) dt = (-1)^n k^{2n-1} \Gamma(n+1) \Gamma(n+1/k) Q_{kn}(z).$$

Let $z = \frac{1}{t} \frac{y}{k^{1/k}}$. Then $D_k^k = kt^k \frac{d}{dy} \frac{1}{y^{k-2}} \frac{d}{dy}$. The left hand side reduces to

$$\begin{aligned} \int_0^\infty e^{-(1-s^2/k)t^2} t^{kn} k^n D_k^{kn} (e^{-s^2 t^2}) dt \\ = \frac{\Gamma(n+1/k)}{\Gamma(1/k)} \int_0^\infty e^{-t^2} t^{kn} k^{2n} \left\{ 1 - \frac{n}{1! (1/k)} z^k t^k + \frac{n(n-1)}{2! (1/k)(1/k+1)} z^{2k} t^{2k} - \dots \right\} dt \end{aligned}$$

which on integration becomes equal to

$$\frac{k^{2n-1} \{\Gamma(n+1/k)\}^2}{\Gamma(1/k)} {}_2F_1(-n, n+1/k; 1/k; z^k) = (-1)^n k^{2n-1} \Gamma(n+1) \Gamma(n+1/k) Q_{kn}(z).$$



We shall next show that

$$Q_{kn}(z) = \frac{1}{1.(k+1)(2k+1)\dots(n-1k+1)n! k^n} D_k^{kn} \frac{1}{(u^k + z^k)^{1/k}},$$

where u^k is to be replaced by $(1-z^k)$ after differentiation.

We know that

$$\frac{1}{(u^k + z^k)^{1/k}} = \left\{ \frac{1}{z} - \frac{(1/k)u^k}{1! z^{k+1}} + \frac{(1/k)(1/k+1)u^{2k}}{2! z^{2k+1}} - \dots + (-1)^r \frac{1/k \dots (1/k+r-1)}{r!} \frac{u^{kr}}{z^{kr+1}} + \dots \right\}$$

Also

$$D_k^{kn} \frac{1}{z^{kr+1}} = \frac{(r+1/k) \dots (r+n-1+1/k)(r+1) \dots (r+n) k^{2n}}{z^{kr+n+1}}.$$

Hence

$$D_k^{kn} \frac{1}{(u^k + z^k)^{1/k}} = 1.(k+1) \dots (n-1k+1)(n)! k^n Q_{kn}(z)$$

on slight reduction.

In the same way we can prove that

$$(1+x^k)^{n+1/k} D_k^{kn} \frac{1}{(1+x^k)^{1/k}} = 1.(k+1)(2k+1) \dots (n-1k+1)(n)! k^n Q_{kn}\left(\frac{x}{(1+x^k)^{1/k}}\right).$$

Let us now suppose that x, y, z are independent variables, so that

$$x^k + y^k + z^k = r^k; \quad z = \mu r.$$

By repeated application of the operator

$$D_k^k = \frac{d}{dz} \frac{1}{z^{k-1}} \frac{d}{dz},$$

it is not difficult to prove inductively that

$$Q_{km}^{(\mu)} = \frac{\Gamma(1/k)^{km+1}}{\Gamma(m+1)\Gamma(m+1/k)k^{2m}} D_k^{km} \left(\frac{1}{r} \right).$$

In conclusion, I wish to express my thanks to Dr. S. C. Mitra for his guidance and help in the preparation of this paper.

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LINEARIZED SUPERSONIC FLOWS AROUND A BODY OF REVOLUTION

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For the flow of a compressible fluid in three dimensions the hypothesis of the existence of potential flow, though not exact, can be accepted as a good approximation. This hypothesis depends on the fact that generally the flow wets the body, that is, the deviation of the stream-line from the direction of the undisturbed flow is small and the variations of the velocity components with respect to the undisturbed velocity are small. If the body has its axis parallel to the flow, the potential motion can be obtained from the potential of a source (or sink) distribution along the axis of the body. The position of the sources and the law of variation of the strength of the sources depend on the shape of the body. The problem is thus reduced to determination of the source distribution as a function of the shape of the body. In this paper a source distribution of constant intensity has been taken along the axis and the shape of the body has been determined for various values of the Mach number and of the intensity of the source.

For a body of revolution, take the x -axis along the axis of the body and the y -axis normal to the x -axis in the meridian plane in the direction of the radius of every circular cross section of the body. If V be the velocity of the undisturbed stream in the direction of x -axis, the components of velocity at any point can be defined as:

$$\left. \begin{aligned} u &= V + u' = V + \frac{\partial \phi'}{\partial x} \\ v &= v' = \frac{\partial \phi'}{\partial y} \end{aligned} \right\} \quad (1)$$

where ϕ' is the potential function which defines the variation of the flow generated by the presence of the body.

If $\phi = \phi(x, y)$ be the potential function of the flow, the differential equation of the potential flow can be written in a simplified form. The simplifying hypotheses are usually based on the assumption that the variations of some of the components of the velocity are small in comparison with the speed of sound a . On this assumption the differential equation for ϕ is (Ferri, 1949)

$$(1 - M^2) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{1}{y} \frac{\partial \phi}{\partial y} = 0, \quad (2)$$

where $M = (V/a)$ is the Mach number of the undisturbed stream. Since

$$\left. \begin{aligned} \frac{\partial \phi}{\partial x} &= u = V + \frac{\partial \phi'}{\partial x} \\ \frac{\partial \phi}{\partial y} &= v = \frac{\partial \phi'}{\partial y} \end{aligned} \right\}, \quad (3)$$

the function ϕ' also satisfies the equation

$$(1-M^2) \frac{\partial^2 \phi'}{\partial x^2} + \frac{\partial^2 \phi'}{\partial y^2} + \frac{1}{y} \frac{\partial \phi'}{\partial y} = 0. \quad (4)$$

The solution of equation (4) is given by the expression (Ferri, 1949)

$$\phi' = - \int_0^{\xi_1} \frac{f(\xi) d\xi}{\{(x-\xi)^2 - B^2 y^2\}^{\frac{1}{2}}}, \quad (5)$$

where $B^2 = M^2 - 1$ and $f(\xi)$ defines the source distribution which will produce a flow that satisfies the boundary conditions. For supersonic flow $M > 1$, the value of the integrand can be imaginary and hence integration should be carried out only over the real values of the integrand and zero is to be substituted for the imaginary values. The value of the upper limit ξ_1 of the integral represents the end of the source phenomenon. Now the integrand in (5) becomes imaginary when

$$(x-\xi)^2 - B^2 y^2 < 0, \text{ or when } \xi > x - By.$$

Therefore the limits of the integral are $\xi = 0$ and $\xi = x - By$. Let us consider a source distribution of constant intensity. Taking $f(\xi) = C$ and writing $\xi = x - By \cosh z$, we get from (5),

$$\phi' = -C \cosh^{-1}(x/By). \quad (6)$$

Therefore

$$\left. \begin{aligned} u &= V - \frac{C}{(x^2 - B^2 y^2)^{\frac{1}{2}}} \\ v &= \frac{Cx}{y(x^2 - B^2 y^2)^{\frac{1}{2}}} \end{aligned} \right\} \quad (7)$$

Now at any point on the surface of the body, the velocity must be along the tangent to the body, hence the boundary condition is given by

$$\frac{dy}{dx} = \frac{v}{u}. \quad (8)$$

Therefore the differential equation for the boundary is

$$\frac{dy}{dx} = \frac{Cx}{y\{V(x^2 - B^2 y^2)^{\frac{1}{2}} - C\}}. \quad (9)$$

The solution of this equation is

$$y^2 = 2 \frac{C}{V} \left[\frac{C}{V} (B^2 + 1) \log \left\{ 1 - \frac{V(x^2 - B^2 y^2)^{\frac{1}{2}}}{C(B^2 + 1)} \right\} + (x^2 - B^2 y^2)^{\frac{1}{2}} \right], \quad (10)$$

$$\text{or } y^2 = 2 \frac{C}{V} \left[\frac{C}{V} (B^2 + 1) \log \left\{ \frac{C}{V} (B^2 + 1) - (x^2 - B^2 y^2)^{\frac{1}{2}} \right\} + (x^2 - B^2 y^2)^{\frac{1}{2}} \right]. \quad (11)$$

For a sink distribution, the corresponding solutions are

$$y^2 = 2 \frac{C}{V} \left[\frac{C}{V} (B^2 + 1) \log \left\{ 1 + \frac{V(x^2 - B^2 y^2)^{\frac{1}{2}}}{C(B^2 + 1)} \right\} - (x^2 - B^2 y^2)^{\frac{1}{2}} \right], \quad (12)$$

$$\text{or } y^2 = 2 \frac{C}{V} \left[\frac{C}{V} (B^2 + 1) \log \left\{ \frac{C}{V} (B^2 + 1) + (x^2 - B^2 y^2)^{\frac{1}{2}} \right\} - (x^2 - B^2 y^2)^{\frac{1}{2}} \right]. \quad (13)$$

Equations (10) and (12) correspond to the case when the origin is at the vertex of the body and (11) and (13) to the case when the origin is not on the body but in front of it.

It is found by numerical analysis that no boundary of the form (10) or (12) exists for any value of C/V and of M . The obvious conclusion is that the source distribution must start ahead of the body.

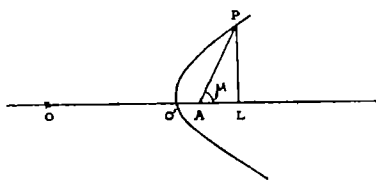


FIG 1

Thus if OO' be the axis of the body, i.e., x -axis, O' the vertex of the body, the origin must be at some point O .

At O' , $y = 0$, therefore $\xi = x$, i.e., the point O' is influenced by the source distribution extending from O to O' . For any other point P on the boundary,

$$\xi = x_P - B y_P = x_P - y_P \cot \mu = OA,$$

μ being the Mach angle, $B = (M^2 - 1)^{\frac{1}{2}} = (\operatorname{cosec}^2 \mu - 1)^{\frac{1}{2}} = \cot \mu$. Therefore the sources distributed along OA will influence the point P . Similarly for any point in the liquid.

We now find the boundary given by (11). The flow around this boundary is produced by a source distribution along the axis over a limited segment. The boundary is given by

$$x^2 - B^2 y^2 = b^2, \quad (14)$$

with

$$y^2 = 2 \frac{C}{V} \left[\frac{C}{V} M^2 \log \left(\frac{C}{V} M^2 - b \right) + b \right], \quad (15)$$

where

$$M > 1, \quad b < \frac{C}{V} M^2 \text{ but } \geq 0.$$

The negative values of b correspond to the sink distribution, note equation (13).

Also since $dx/dy = 0$ when $y = 0$ and $(x^2 - B^2 y^2)^{\frac{1}{2}} = C/V$, we see that x will be maximum when $b = C/V$. At this point the base of the body will be situated.

The solutions of equations (14) and (15) have been carried out by numerical analysis. Since the variations of the velocity components from the undisturbed stream are small C must be small compared with V . In the following tables, values of x and of y have been tabulated corresponding to different values of C/V and of M

$$C/V = 0.2, M = 3$$

| | | | | | | | |
|-----|------|------|------|------|------|------|------|
| b | 0 | 0.20 | 0.50 | 1.00 | 1.20 | 1.30 | 1.32 |
| x | 1.84 | 1.84 | 1.83 | 1.70 | 1.53 | 1.36 | 1.32 |
| y | 0.65 | 0.63 | 0.62 | 0.48 | 0.33 | 0.14 | 0 |

When $y = 0$, x is slightly different from 1.32. The vertex of the body is at a distance 1.32 from the origin and the height of the body is nearly 0.52

$$C/V = 0.2, M = 4$$

| | | | | | | | |
|-----|------|------|------|------|------|------|------|
| b | 0 | 0.20 | 1.00 | 2.00 | 2.50 | 2.70 | 2.75 |
| x | 4.72 | 4.72 | 4.72 | 4.41 | 3.80 | 3.00 | 2.75 |
| y | 1.22 | 1.22 | 1.20 | 1.01 | 0.74 | 0.34 | 0 |

Distance of the vertex from the origin = 2.75 nearly, height = 1.97

$$C/V = 0.5, M = 2$$

| | | | | | |
|-----|------|------|------|------|------|
| b | 0 | 0.50 | 1.00 | 1.50 | 1.60 |
| x | 2.04 | 2.04 | 2.00 | 1.61 | 1.60 |
| y | 1.18 | 1.15 | 1.00 | 0.34 | 0 |

Distance of the vertex from the origin = 1.6, height = 0.44.

$$C/V = 0.5, M = 3$$

| | | | | | | | |
|-----|------|------|------|------|------|------|------|
| b | 0 | 0.50 | 1.00 | 2.00 | 3.00 | 4.00 | 4.10 |
| x | 7.36 | 7.36 | 7.35 | 7.28 | 6.80 | 4.72 | 4.10 |
| y | 2.60 | 2.59 | 2.58 | 2.47 | 2.19 | 0.94 | 0 |

Distance of the vertex from the origin = 4.1, height = 3.26.

$$C/V = 0.5, M = 4$$

| | | | | | | | | | |
|-----|-------|-------|-------|-------|-------|-------|------|------|------|
| b | 0 | 0.50 | 2.00 | 4.00 | 6.00 | 7.00 | 7.50 | 7.60 | 7.62 |
| x | 15.80 | 15.80 | 15.70 | 15.50 | 14.50 | 12.40 | 9.20 | 7.80 | 7.62 |
| y | 4.08 | 4.08 | 4.04 | 3.87 | 3.40 | 2.64 | 1.40 | 0.52 | 0 |

Distance of the vertex from the origin = 7.62 nearly, height = 8.18.

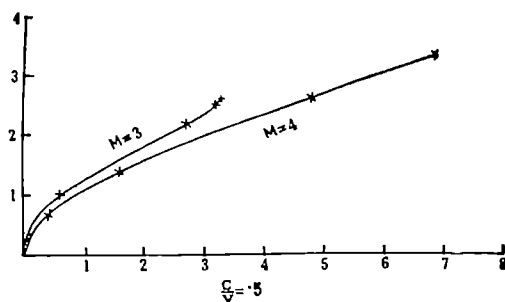


FIG 2

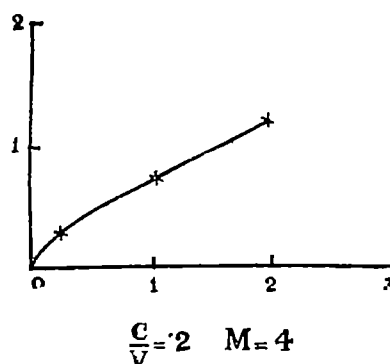


FIG 3

The figures 1 and 2 give only one-half of a meridian section. The surface is obtained by revolving this section about the x axis. For convenience all the surfaces have been drawn having the same vertex

From these figures we see that for small values of M the surface is very stiff or blunt at the vertex and as M increases the surface becomes more elongated and the nose becomes sharper and the large portion, except near the vertex, is more or less straight. There is a slight bend at the base.

We also note that as C/V or M increases, the extension of the source distribution as well as the height of the body increases. We conclude that a source (sink) distribution of constant intensity extending along the axis over a limited length in front of the body can produce a supersonic potential motion of a compressible fluid around a body of revolution whose shape can be easily determined.

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SOME PROPERTIES OF THE SKEWNESS OF DISTRIBUTION OF THE GENERATORS OF A RULED SURFACE

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1. In a recent paper, V. Ranga Chariar (1945) has defined the skewness of distribution of the generators of a ruled surface, and has discussed some of its properties. The object of this paper is to obtain some further properties of the skewness of distribution and to extend the idea of skewness of distribution to the ruled surfaces through a line of a rectilinear congruence.

2. The equations to a developable surface are given by

$$\xi = x + l_1 u, \quad \eta = y + m_1 u, \quad \zeta = z + n_1 u,$$

where l_1, m_1, n_1 are the direction cosines of the tangent to the edge of regression.

Therefore the skewness of distribution μ is given by

$$\mu = [d, d', d''] / [d']^3$$

$$= \frac{\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2/\rho & m_2/\rho & n_2/\rho \\ \frac{1}{\rho} \left(\frac{l_3}{\sigma} - \frac{l_1}{\rho} \right) - \frac{l_2}{\rho^2} \rho' & \dots & \dots \end{vmatrix}}{\left(\frac{1}{\rho^3} \right)^{3/2}} = \frac{1}{\rho^4 \sigma} \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \left/ \left(\frac{1}{\rho^3} \right)^{3/2} \right. = \frac{1}{\rho^2 \sigma} \rho^3 = \frac{\rho}{\sigma}.$$

Therefore if μ is constant, the edge of regression of the developable surface is a helix.

Also

$$\xi_1 = l_1; \quad \eta_1 = m_1; \quad \zeta_1 = n_1,$$

$$\xi_2 = l_1 + u(l_2/\rho); \quad \eta_2 = m_1 + u(m_2/\rho); \quad \zeta_2 = n_1 + u(n_2/\rho),$$

where suffices 1 and 2 denote partial differentiations with regard to u , and s (the arc of the edge of regression) respectively. Therefore

$$E = 1, \quad G = 1 + \frac{u^2}{\rho^3}, \quad F = 1,$$

$$\xi_{11} = 0, \quad \eta_{11} = 0, \quad \zeta_{11} = 0,$$

$$\xi_{12} = \frac{l_2}{\rho}, \quad \eta_{12} = \frac{m_2}{\rho}, \quad \zeta_{12} = \frac{n_2}{\rho},$$

$$\xi_{22} = \frac{l_2}{\rho} + \frac{u}{\rho} \left(\frac{l_2}{\sigma} - \frac{l_1}{\rho} \right) - u \frac{l_2}{\rho^2} \rho', \text{ etc.}$$

$$\therefore L = 0, \quad M = 0, \quad N = u/\rho\sigma.$$

The first curvature of the developable surface is

$$J = \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{EN - 2FM + GL}{EG - F^2} = \frac{u \cdot \rho^2}{\rho\sigma u^2} = \frac{\rho}{u\sigma} = \frac{\mu}{u}.$$

But in the case of a developable surface one of the principal curvatures is zero, say $1/\rho_2 = 0$. Then

$$\frac{1}{\rho_1} = \frac{\mu}{u}, \text{ or } \rho_1 = \frac{u}{\mu}$$

Hence in the case of a developable surface, the principal radius at a point distant u from the edge of regression, is u/μ where μ is the skewness of distribution.

3. We now proceed to find the skewness of distribution of the ruled surface formed by the binormals to a given twisted curve. The equations of this ruled surface are

$$\xi = x + l_3 u, \quad \eta = y + m_3 u, \quad \zeta = z + n_3 u,$$

where l_3, m_3, n_3 are the direction cosines of the binormal.

$$\begin{aligned} \therefore \mu &= \frac{\begin{vmatrix} l_3 & m_3 & n_3 \\ l'_3 & m'_3 & n'_3 \\ l''_3 & m''_3 & n''_3 \end{vmatrix}}{(l_3'^2 + m_3'^2 + n_3'^2)^{3/2}} = \frac{\begin{vmatrix} l_3 & m_3 & n_3 \\ -l_2/\sigma & -m_2/\sigma & -n_2/\sigma \\ -\frac{1}{\sigma}\left(\frac{l_3}{\sigma} - \frac{l_1}{\rho}\right) + \frac{l_2}{\sigma^2}\sigma' & \dots & \dots \end{vmatrix}}{\left(\frac{1}{\sigma^2}\right)^{3/2}} \\ &= \frac{1}{\rho\sigma^2} \frac{\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}}{\frac{1}{\sigma^3}} = \frac{1}{\rho\sigma^2} \cdot \sigma^3 = \frac{\sigma}{\rho}. \end{aligned}$$

Hence the skewness of distribution of the ruled surface formed by the binormals to a given twisted curve is equal to the reciprocal of the skewness of distribution of the developable surface formed by the tangents to the given curve.

4. In the same manner, the skewness of distribution μ of the ruled surface formed by the principal normals to a given twisted curve is given by, (l_2, m_2, n_2) being the direction cosines of the principal normal to the given curve,

$$\begin{aligned} \mu &= \frac{\begin{vmatrix} l_2 & m_2 & n_2 \\ \frac{l_3}{\sigma} - \frac{l_1}{\rho} & \dots & \dots \\ -\frac{l_2}{\sigma^2} - \frac{l_3}{\sigma^2}\sigma' - \frac{l_2}{\rho^2} + \frac{l_1}{\rho^2}\rho' & \dots & \dots \end{vmatrix}}{\left(\frac{1}{\sigma^2} + \frac{1}{\rho^2}\right)^{3/2}} \\ &= \frac{\begin{vmatrix} l_2 & m_2 & n_2 \\ \frac{l_3}{\sigma} - \frac{l_1}{\rho} & \frac{m_3}{\sigma} - \frac{m_1}{\rho} & \frac{n_3}{\sigma} - \frac{n_1}{\rho} \\ \frac{l_1}{\rho^2}\rho' - \frac{l_3}{\sigma^2}\sigma' & \frac{m_1}{\rho^2}\rho' - \frac{m_3}{\sigma^2}\sigma' & \frac{n_1}{\rho^2}\rho' - \frac{n_3}{\sigma^2}\sigma' \end{vmatrix}}{\left(\frac{1}{\sigma^2} + \frac{1}{\rho^2}\right)^{3/2}} \\ &= \left\{ \frac{\rho'}{\sigma\rho^2} \begin{vmatrix} l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \\ l_1 & m_1 & n_1 \end{vmatrix} + \frac{\sigma'}{\rho\sigma^2} \begin{vmatrix} l_2 & m_2 & n_2 \\ l_1 & m_1 & n_1 \\ l_3 & m_3 & n_3 \end{vmatrix} \right\} / \left(\frac{1}{\sigma^2} + \frac{1}{\rho^2}\right)^{3/2} = \left\{ \frac{1}{\rho\sigma} \left(\frac{\rho'}{\rho} - \frac{\sigma'}{\sigma} \right) \right\} / \left(\frac{1}{\sigma^2} + \frac{1}{\rho^2}\right)^{3/2} \end{aligned}$$

or

$$\mu = \rho\sigma(\rho'\sigma - \sigma'\rho)/(\rho^2 + \sigma^2)^{3/2}.$$

If the given twisted curve be a helix, $\rho/\sigma = \text{constant}$, and therefore

$$\sigma\rho' - \rho\sigma' = 0. \quad \therefore \mu = 0.$$

Hence we get the result that *for the ruled surface generated by the principal normals to a helix, the skewness of distribution vanishes, which can be seen to be true from purely geometrical considerations.*

5. Again, the system of curved asymptotic lines of the ruled surface,

$$x = p + lu, \quad y = q + mu, \quad z = r + nu,$$

is given by (Ram Behari, 1946)

$$\frac{du}{dv} + \frac{\lambda}{2\delta} + \frac{\mu}{2\delta}u + \frac{\nu}{2\delta}u^2 = 0,$$

where

$$\begin{aligned} \lambda &= \sum l(q'r'' - r'q''), & \nu &= \sum l(m'n'' - n'm''), \\ \mu &= \sum l\{(m'r'' - m''r') + (q'n' - n'q'')\}, & \delta &= -[p', l, l']. \end{aligned}$$

This is Riccati's equation. From definition of the skewness of distribution $\bar{\mu}$, it follows that

$$\nu = \bar{\mu}(l'^2 + m'^2 + n'^2)^{3/2}.$$

If $\bar{\mu} = 0$, then $\nu = 0$, and the equation of the curved asymptotic lines becomes

$$\frac{du}{dv} + \frac{\lambda}{2\delta} + \frac{\mu}{2\delta}u = 0.$$

Further if $\nu = 0$, the generators remain parallel to a fixed plane. Taking this fixed plane as the plane of xy , the equation of the surface can be written, without loss of generality, in the form

$$x = u, \quad y = v + f(v)u, \quad z = F(v).$$

Here

$$\delta = -f'F', \quad \mu = f'F'' - f''F', \quad \lambda = F''.$$

Therefore the equation giving the curved asymptotic lines becomes

$$\frac{du}{dv} - \frac{F'}{2f'F'} - \frac{1}{2f'F'}(f'F'' - f''F')u = 0,$$

whose solution is

$$u = \sqrt{\left(\frac{F'}{f'}\right)} \left[\int \frac{F''}{2(f'F')^{3/2}} dv + \text{constant} \right].$$

Hence the curved asymptotic lines of a ruled surface whose skewness of distribution is zero can be obtained by two quadratures.

6. We now proceed to find the properties of the skewness of distribution μ , for the ruled surfaces through a line of a rectilinear congruence.

Let $r(X, Y, Z)$ be the direction cosines of a ray of the congruence meeting a curve C on the director surface. These rays form a ruled surface Σ . Let ds be the linear element of the curve C and $d\sigma$ the linear element of the spherical representation of the corresponding ruled surface. Let $d = r$. Then

$$d' = \frac{dr}{d\sigma} \frac{d\sigma}{ds} = t\sigma',$$

where t is the unit tangent to the spherical representation, and dashes denote differentiation with respect to the arc length of the directrix C of the ruled surface Σ .

Differentiating again

$$d' = \frac{dt}{d\sigma} \sigma'^2 + t\sigma'' = kn\sigma'^2 + t\sigma'',$$

where kn is the vector curvature of the spherical representation

$$\begin{aligned} \therefore \mu &= [d, d', d''] / |d'|^3 = [r, t\sigma', kn\sigma'^2 + t\sigma''] / |d'|^3 \\ &= [r, t, kn] \sigma'^3 / |t|^3 \sigma'^3 = [r, t, kn], \end{aligned}$$

or

$$\mu = \begin{vmatrix} X & Y & Z \\ X_1 \frac{du}{d\sigma} + X_2 \frac{dv}{d\sigma} & \dots & \dots \\ X_{11} \left(\frac{du}{d\sigma}\right)^2 + 2X_{12} \frac{du}{d\sigma} \frac{dv}{d\sigma} + X_{22} \left(\frac{dv}{d\sigma}\right)^2 + X_2 \frac{d^2v}{d\sigma^2} & \dots & \dots \end{vmatrix}$$

The converse problem *i.e.*, given a specified μ , to find the curve C or the spherical representation of the corresponding ruled surface cannot in general be solved; for the equation giving μ is only one ordinary second order differential equation in two dependent variables, and therefore the differential equation is indeterminate. If however μ is zero, then

$$[r, t, kn] = 0$$

i.e., the vector curvature lies in the normal plane through t and coincides with the normal to the surface. Therefore the spherical representations are geodesics on the unit sphere, *i.e.*, are great circles through the point.

7. We now proceed to find the skewness of distribution of the five families of ruled surfaces through a line of a rectilinear congruence.

Let the spherical representations of the principal surfaces be taken as parametric curves. Then (Ram Behari, 1946, p. 52)

$$F = 0, \quad f + f' = 0 \quad \text{and} \quad d\sigma^2 = Edu^2 + Gdv^2.$$

For the principal surface represented by the parametric curve $u = \text{constant}$, $d\sigma = \sqrt{G} dv$ and therefore

$$t = \frac{dX}{d\sigma}, \frac{dY}{d\sigma}, \frac{dZ}{d\sigma} = \frac{X_2}{\sqrt{G}}, \frac{Y_2}{\sqrt{G}}, \frac{Z_2}{\sqrt{G}};$$

$$kn = \frac{dt}{d\sigma} = \frac{1}{\sqrt{G}} \frac{d}{dv} \left(\frac{X_2}{\sqrt{G}} \right), \dots = \frac{GX_{22} - G_2X_2}{G^2}, \frac{GY_{22} - G_2Y_2}{G^2}, \frac{GZ_{22} - G_2Z_2}{G^2}.$$

Therefore

$$\mu_1 = \begin{vmatrix} X & Y & Z \\ \frac{X_2}{\sqrt{G}} & \frac{Y_2}{\sqrt{G}} & \frac{Z_2}{\sqrt{G}} \\ \frac{GX_{22} - G_2X_2}{G^2} & \frac{GY_{22} - G_2Y_2}{G^2} & \frac{GZ_{22} - G_2Z_2}{G^2} \end{vmatrix} = G^{-3/2} \begin{vmatrix} X & Y & Z \\ X_2 & Y_2 & Z_2 \\ X_{22} & Y_{22} & Z_{22} \end{vmatrix}.$$

Similarly the skewness of distribution μ_2 for the principal surface represented by the parametric curve $v = \text{constant}$ is given by

$$\mu_2 = E^{-3/2} \begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_{11} & Y_{11} & Z_{11} \end{vmatrix}$$

With the above choice of parametric curves, the directions of the spherical representations of the developable surfaces are given by

$$\begin{vmatrix} Edu & Gdv \\ edu + fdv & -fdv + gdv \end{vmatrix} = 0$$

or

$$Efdv^2 + (eG - Eg)dudv + fGdv^2 = 0$$

or

$$Ef\lambda^2 + (eG - Eg)\lambda + fG = 0, \text{ where } \lambda = du/dv.$$

An element $d\sigma$ of the spherical representation is given by

$$\begin{aligned} d\sigma^2 &= Edu^2 + Gdv^2, \text{ since } F = 0, \\ &= (E\lambda^2 + G)dv^2 \end{aligned}$$

or

$$d\sigma = (E\lambda^2 + G)^{1/2} dv$$

$$\therefore \frac{dX}{d\sigma} = \frac{dX}{(E\lambda^2 + G)^{1/2} dv} = \frac{X_2}{(E\lambda^2 + G)^{1/2}}.$$

Therefore

$$\begin{aligned} \frac{d^3X}{d\sigma^3} &= \frac{1}{(E\lambda^2 + G)^{3/2}} \cdot \frac{d}{dv} \left\{ \frac{X_2}{(E\lambda^2 + G)^{1/2}} \right\} \\ &= \frac{1}{(E\lambda^2 + G)^{3/2}} \cdot \frac{(E\lambda^2 + G)^{1/2} X_{22} - X_2 \cdot \frac{1}{2} (E\lambda^2 + G)^{-1/2} (E_2\lambda^2 + 2E\lambda\lambda_2 + G_2)}{(E\lambda^2 + G)} \\ &= \frac{(E\lambda^2 + G)X_{22} - \frac{1}{2}X_2(E_2\lambda^2 + 2E\lambda\lambda_2 + G_2)}{(E\lambda^2 + G)^{3/2}}. \end{aligned}$$

Therefore μ , the skewness of distribution of the developable surface

$$= (E\lambda^2 + G)^{-3/2} \begin{vmatrix} X & Y & Z \\ X_2 & Y_2 & Z_2 \\ X_{22} & Y_{22} & Z_{22} \end{vmatrix}.$$

If the congruence be normal, then $f = f'$ and $\therefore f = f' = 0$, since $f = -f'$. Therefore the equation giving λ becomes

$$(eG - Eg)\lambda = 0$$

Therefore $\lambda = 0$ rejecting the case $e/E = \bar{g}/G$ when the congruence is isotropic.

In this case $(E\lambda^2 + G)^{-3/2} = G^{-3/2}$ and we find that the μ for the principal surface coincides with μ for the developable surface.

Incidentally we get the result that if the skewness of distribution for the principal surfaces through a line of a rectilinear congruence is the same as that for the developable surfaces through that line, the congruence is normal, which gives us a necessary and sufficient condition for a rectilinear congruence to be normal.

In the same way it can be shewn that, when the congruence is normal, the skewness of distribution of the characteristic ruled surfaces through a line of the congruence is the same as that of the ruled surfaces whose spherical representation are minimal lines.

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SOME PROPERTIES OF THE WHITTAKER TRANSFORM

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1. The integral

$$\varphi(p) = p \int_0^{\infty} (2xp)^{-\frac{1}{2}} W_{k,m}(2px) f(x) dx \quad (1)$$

is the generalisation of the Laplace Integral and the new transform is known as Whittaker transform, according to Dr. Varma (1947). $\varphi(p)$ is called the *image* of $f(x)$ and $f(x)$ the *original* of $\varphi(p)$, and I (1949) have given the symbolic notation to integral (1) as

$$\varphi(p) \stackrel{k}{\underset{m}{\rightleftharpoons}} f(x).$$

For $k = \frac{1}{2}$, $m = \pm \frac{1}{2}$, the above transform reduces to Laplace Transform, due to

$$(2xp)^{-\frac{1}{2}} W_{\frac{1}{2}, \pm \frac{1}{2}}(2px) \equiv e^{-px}.$$

There are other particular cases of this transform, due to the fact that for certain values of the parameters, k and m , Whittaker function reduces to other well known functions.

This paper is an attempt to establish some new properties of this new transform, the analogues of which do not exist in the case of Operational Calculus. The properties of Whittaker and Parabolic Cylinder functions have been utilised in deriving these properties.

2. **Theorem I.** If

$$\varphi_r(p) \stackrel{k+r}{\underset{m}{\rightleftharpoons}} f(x)$$

then

$$\sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{1}{\alpha} - 1 \right)^r \varphi_r(p) = p^{\alpha k} \int_0^{\infty} (2px)^{-\frac{1}{2}} e^{xp(1-\alpha)} W_{k,m}(2px) f(x) dx$$

provided $\mathbf{R}(\mu \pm m + 5/4) > 0$, where $f(x) = O(x^{\mu})$ for small x , $\alpha > \frac{1}{2}$ and $x^{-\frac{1}{2}} W_{k,m}(2px) f(x)$ is bounded for $x \geq 0$, $\mathbf{R}\{(2\alpha-1)p\} > p_0 > 0$ and the series on the left hand side is uniformly convergent.

Proof: We have

$$\varphi_r(p) = p \int_0^{\infty} (2xp)^{-\frac{1}{2}} W_{k+r,m}(2px) f(x) dx \quad (1)$$

Multiplying by $\frac{1}{r!} \left(\frac{1}{\alpha} - 1 \right)^r$ and taking the sum from zero to infinity and using the result of Goldstein (1932, p. 112)

$$\alpha^k e^{-\frac{1}{2}\alpha x} W_{k,m}(\alpha x) = e^{-\frac{1}{2}\alpha x} \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{1}{\alpha} - 1\right)^r W_{k+r,m}(x) \quad (2)$$

where α is positive, we have

$$\sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{1}{\alpha} - 1\right)^r \varphi_r(p) = p \alpha^k \int_0^{\infty} (2xp)^{-\frac{1}{2}} e^{xp(1-\alpha)} W_{k,m}(2px) f(x) dx \quad (3)$$

The change of order of summation and integration is permissible due to the uniform and absolute convergence of the series and the integral.

3. Example: Let (Bose, 1949, p. 18)

$$f(x) = x^n e^{-qx} \frac{\Gamma(n+m+5/4) \Gamma(n-m+5/4)}{2(2p)^n \Gamma(n-k-r+7/4)} {}_2F_1 \left\{ \begin{matrix} n+m+5/4, & n-m+5/4 \\ n-k-r+7/4 \end{matrix} ; \frac{1}{2} - \frac{q}{2p} \right\},$$

$$\mathbf{R}(n \pm m + 5/4) > 0, \quad |p| > |q| \text{ and } \mathbf{R}(p) > 0.$$

Hence by Theorem I, we get

$$\int_0^{\infty} (2xp)^{n-\frac{1}{2}} W_{k,m}(2px) e^{xp(1-\alpha)-qx} dx = \frac{\Gamma(n+m+5/4) \Gamma(n-m+5/4)}{(2p)^{n+1} \alpha^k} \sum_{r=0}^{\infty} \frac{(1/\alpha - 1)^r}{r! \Gamma(n-k-r+7/4)}$$

$$\times {}_2F_1 \left\{ \begin{matrix} n+m+5/4, & n-m+5/4 \\ n-k-r+7/4 \end{matrix} ; \frac{1}{2} - \frac{q}{2p} \right\},$$

$$\mathbf{R}(n \pm m + 5/4) > 0, \quad \alpha > \frac{1}{2}, \quad \mathbf{R}\{p(2\alpha - 1) + q\} > 0 \text{ and } |p| > |q|.$$

4. Theorem II. If

$$\varphi_s(p) \stackrel{s+\frac{1}{2}}{=} f(x)$$

then

$$\psi(p) = \frac{1+h}{(2p)^{\frac{1}{2}}} \sum_{s=0}^{\infty} \frac{(-h)^s}{s!} \varphi_s(p),$$

where

$$\psi(p) \doteq (x/a)^{\frac{1}{2}} f(x/a) \quad \text{and} \quad a = \frac{1+h}{1-h},$$

provided $\mathbf{R}(\mu + 5/4) > 0$, $f(x) = O(x^\mu)$ for small x , $\mathbf{R}(p) > p_0 > 0$, $|h| < 1$ and the series on the right hand side is convergent.

Proof: We have

$$\varphi_s(p) = p \int_0^{\infty} (2xp)^{-\frac{1}{2}} W_{s+\frac{1}{2},0}(2px) f(x) dx \quad (1)$$

Multiplying by $(-h)^s/s!$ and taking the sum from zero to infinity and using the result (Goldstein, 1982, p. 112)

$$(1-h)^{-1} e^{-\frac{h}{1-h}x} = x^{-\frac{1}{2}} e^{\frac{1}{2}x} \sum_{s=0}^{\infty} \frac{(-h)^s}{s!} W_{s+\frac{1}{2},0}(x), \quad h < 1$$

we get

$$\sum_{s=0}^{\infty} \frac{(-h)^s}{s!} \varphi_s(p) = \frac{p}{1-h} \int_0^{\infty} (2xp)^{\frac{1}{2}} e^{-xp} \left(\frac{1+h}{1-h}\right) f(x) dx = \frac{(2p)^{\frac{1}{2}}}{1+h} \psi(p),$$

where

$$\psi(p) \doteq (x/a)^{\frac{1}{2}} f(x/a) \quad \text{and} \quad a = \frac{1+h}{1+h}.$$

The change of order of integration and summation can be justified as before.

5. *Example.* Let (Bose, 1949, p. 12)*

$$f(x) = x^{n+1} {}_1F_1[\alpha; \beta; -qx] \frac{s+\frac{1}{2}}{0} \frac{\Gamma(\beta)}{2(2p)^n \Gamma(\alpha)} \sum_{r=0}^{\infty} \frac{(-)^r \Gamma(\alpha+r) \Gamma(n+r+5/4) \Gamma(n+r+5/4)}{r! \Gamma(\beta+r) \Gamma(n-s+r+5/4)} \\ \times \left(\frac{q}{2p}\right)^r {}_2F_1\left\{\begin{matrix} n+r+5/4, & n+r+5/4 \\ n-s+r+5/4 \end{matrix}; \frac{1}{2}\right\}, \quad \mathbf{R}(n+5/4) > 0, \quad \mathbf{R}(p) > 0 \quad \text{and} \quad |p| > |q|,$$

and

$$\psi(p) = \frac{\Gamma(n+5/4)}{(ap)^{n+1/2}} {}_2F_1\left\{\begin{matrix} \alpha, & n+5/4 \\ \beta \end{matrix}; -\frac{q}{ap}\right\} \doteq (x/a)^{\frac{1}{2}} f(x/a).$$

Hence by Theorem II, we get, that the sum of the double series is a hypergeometric series,

$$\frac{\Gamma(\alpha) \Gamma(n+5/4)}{a^{n+1/2} \Gamma(\beta)} {}_2F_1\left\{\begin{matrix} \alpha, & n+5/4 \\ \beta \end{matrix}; -\frac{q}{ap}\right\} \\ = \frac{1+h}{2^{n+5/4}} \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-)^{s+r} h^s \Gamma(\alpha+r) \Gamma(n+r+5/4) \Gamma(n+r+5/4)}{s! r! \Gamma(\beta+r) \Gamma(n-s+r+5/4)} \left(\frac{q}{2p}\right)^r \\ \times {}_2F_1\left\{\begin{matrix} n+r+5/4, & n+r+5/4 \\ n-s+r+5/4 \end{matrix}; \frac{1}{2}\right\}, \\ \mathbf{R}(n+5/4) > 0, \quad \mathbf{R}(p) > p_0 > 0, \quad |h| < 1 \quad \text{and} \quad |p| > |q|.$$

6. **Theorem III.** If

$$\varphi_{2r}(p) \doteq \frac{\frac{1}{2}\mu+r-\frac{1}{2}}{\frac{1}{2}} f(x),$$

then

$$\int_0^{\infty} e^{px(1-\alpha^2)} D_{\mu-1}\{2\alpha(px)^{\frac{1}{2}}\} f(x) dx = \frac{2^{\frac{1}{2}\mu-\frac{1}{2}}}{\alpha^{\frac{1}{2}} p} \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{1}{\alpha^2} - 1\right)^r \varphi_{2r}(p)$$

provided $\mathbf{R}(\lambda) > -1$, where $f(x) = O(x^\lambda)$ for small x , $\mathbf{R}\{(2\alpha^2-1)p\} > p_0 > 0$, $\alpha^2 > \frac{1}{2}$ and the series on the right hand side is convergent.

Proof: We have

$$\varphi_{2r}(p) = 2^{-\frac{1}{2}(\mu+2r-1)} p \int_0^{\infty} D_{\mu+2r-1}\{2(xp)^{\frac{1}{2}}\} f(x) dx. \quad (1)$$

Multiplying by $\frac{1}{r!} \left(\frac{1}{\alpha^2} - 1\right)^r$ and taking the sum from zero to infinity and using the result* (Goldstein, 1932, p. 116)

* The convergence of the series follows by considering the behaviour

$$F[a+r, b+s; c+r; x] < \frac{(c)_r}{(A)_r} (1-x)^{-r} F[a, b; c; x]$$

where $A = \min(a, b, c)$.

$$\alpha^\mu e^{-\frac{1}{2}\alpha^2} D_{\mu-1}(\bar{\alpha}x) = e^{-\frac{1}{2}\alpha^2} \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{1}{\alpha^2} - 1 \right)^r D_{\mu+2r-1}(x), \quad \alpha > 0$$

we get

$$2^{\frac{1}{2}\mu-\frac{1}{2}} \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{1}{\alpha^2} - 1 \right)^r \varphi_{2r}(p) = \alpha^\mu p \int_0^\infty e^{px(1-\alpha^2)} D_{\mu-1}\{2\alpha(xp)^{\frac{1}{2}}\} f(x) dx \quad (2)$$

provided $\mathbf{R}(\lambda) > -1$, where $f(x) = O(x^\lambda)$ and small x , $\mathbf{R}\{(2\alpha^2-1)p\} > p_0 > 0$, $\alpha^2 > \frac{1}{2}$ and the series on the left hand side is convergent.

The change of order of integration and summation can be justified as before.

7. *Example.* Let (Bose, 1949, p. 14)

$$f(x) = J_0\{2x^{\frac{1}{2}}\}^{\frac{1}{2}\mu+\frac{r-1}{2}} \frac{1}{\pm \frac{1}{2}} \sum_{s=0}^{\infty} \frac{(-)^s \Gamma(s+\frac{3}{2})(2p)^{-s}}{s! \Gamma(s-\frac{1}{2}\mu-r+2)} {}_2F_1\left\{\begin{matrix} s+\frac{3}{2}, s+1 \\ s-\frac{1}{2}\mu-r+2 \end{matrix}; \frac{1}{2}\right\}$$

Hence by Theorem III, we get

$$\begin{aligned} & \int_0^\infty e^{px(1-\alpha^2)} D_{\mu-1}\{2\alpha(xp)^{\frac{1}{2}}\} J_0\{2x^{\frac{1}{2}}\} dx \\ &= \frac{2^{\frac{1}{2}\mu+\frac{1}{2}}}{\alpha^\mu p} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-)^s \Gamma(s+\frac{3}{2})(2s)^{-s}}{r! s! \Gamma(s-\frac{1}{2}\mu-r+2)} \left(\frac{1}{\alpha^2} - 1 \right)^r {}_2F_1\left\{\begin{matrix} s+\frac{3}{2}, s+1 \\ s-\frac{1}{2}\mu-r+2 \end{matrix}; \frac{1}{2}\right\}, \\ & \mathbf{R}\{(2\alpha^2-1)p\} > p_0 > 0 \text{ and } \alpha^2 > \frac{1}{2}. \end{aligned}$$

8. **Theorem IV.** If

$$\varphi_r(p) \frac{\frac{1}{2}\mu+\frac{1}{2}r+\frac{1}{2}}{\pm \frac{1}{2}} f(x),$$

then

$$\int_0^\infty \exp [px - \frac{1}{4}\{2(p x)^{\frac{1}{2}} + \eta\}^2] D_\mu\{2(p x)^{\frac{1}{2}} + \eta\} f(x) dx = \frac{2^{\frac{1}{2}\mu}}{p} \sum_{r=0}^{\infty} \frac{(-)^r 2^{\frac{1}{2}r} \eta^r}{r!} \varphi_r(p)$$

provided $\mathbf{R}(p) > p_0 > 0$, $\mathbf{R}(\lambda) > -1$, where $f(x) = O(x^\lambda)$ for small x and the series on the right hand side is convergent.

Proof: We have

$$\varphi_r(p) = 2^{-\frac{1}{2}(\mu+r)} p \int_0^\infty D_{\mu+r}\{2(p x)^{\frac{1}{2}}\} f(x) dx. \quad (1)$$

Multiplying by $(-)^r \eta^r / r!$ and taking the sum from zero to infinity, and using (Goldstein, 1952, p. 116)

$$e^{-\frac{1}{4}(x+\eta)^2} D_\mu(x+\eta) = e^{-\frac{1}{4}x^2} \sum_{r=0}^{\infty} \frac{(-)^r \eta^r}{r!} D_{\mu+r}(x)$$

we get

$$2^{\frac{1}{2}\mu} \sum_{r=0}^{\infty} \frac{(-)^r 2^{\frac{1}{2}r} \eta^r}{r!} \varphi_r(p) = p \int_0^\infty \exp [px - \frac{1}{4}\{2(p x)^{\frac{1}{2}} + \eta\}^2] D_\mu\{2(p x)^{\frac{1}{2}} + \eta\} f(x) dx$$

provided $\mathbf{R}(\lambda) \geq -1$, where $f(x) = O(x^\lambda)$ for small x , $\mathbf{R}(p) > p_0 \geq 0$ and the series on the left hand side is convergent.

The change of order of integration and summation can be justified as before.

9. Theorem V. *If*

$$\varphi_{n,\lambda}(p) \stackrel{\frac{1}{2}n+\frac{1}{2}}{\pm\frac{1}{2}} x^{-\lambda} f(x),$$

then

$$\varphi_{n+1,\lambda}(p) = (2p)^{\frac{1}{2}} \varphi_{n,\lambda-\frac{1}{2}}(p) - \frac{1}{2} n \varphi_{n-1,\lambda}(p)$$

provided $\mathbf{R}(\mu - \lambda + 1) > 0$, where $f(x) = O(x^\mu)$ for small x and $\mathbf{R}(p) > p_0 > 0$.

Proof: We have

$$\varphi_{n+1,\lambda}(p) = 2^{-\frac{1}{2}(n+1)} p \int_0^\infty D_{n+1}\{2(px)^{\frac{1}{2}}\} x^{-\lambda} f(x) dx. \quad (1)$$

Using (Goldstein, 1932, p. 115)

$$D_{n+1}(x) = x D_n(x) - n D_{n-1}(x)$$

in (1), we get

$$\varphi_{n+1,\lambda}(p) = (2p)^{\frac{1}{2}} \varphi_{n,\lambda-\frac{1}{2}}(p) - \frac{1}{2} n \varphi_{n-1,\lambda}(p). \quad (2)$$

The result (2) shows that there exists a recurrence formula of the *images of any function* $x^{-\lambda} f(x)$, where λ is an arbitrary parameter, for which the integrals and the integrated series involved are uniformly convergent.

10. Example: Let (Bose, 1949, p. 18)

$$x^{-\lambda} f(x) = x^{\nu-\lambda} e^{-qx} \stackrel{\frac{1}{2}n+\frac{1}{2}}{\pm\frac{1}{2}} \frac{\Gamma(\nu-\lambda+\frac{3}{2})\Gamma(\nu-\lambda+1)}{2(2p)^{\nu-\lambda}\Gamma(\nu-\lambda-\frac{1}{2}n+\frac{3}{2})} \cdot {}_2F_1\left\{\begin{matrix} \nu-\lambda+\frac{3}{2}, \nu-\lambda+1 \\ \nu-\lambda-\frac{1}{2}n+\frac{3}{2} \end{matrix}; \frac{1}{2}-\frac{q}{2p}\right\}.$$

Hence by Theorem V, we get

$$\begin{aligned} & \frac{\Gamma(\nu-\lambda-\frac{1}{2}n+\frac{3}{2})}{\Gamma(\nu-\lambda-\frac{1}{2}n+1)} \cdot {}_2F_1\left\{\begin{matrix} \nu-\lambda+\frac{3}{2}, \nu-\lambda+1 \\ \nu-\lambda-\frac{1}{2}n+1 \end{matrix}; \frac{1}{2}-\frac{q}{2p}\right\} \\ &= (\nu-\lambda+1) \cdot {}_2F_1\left\{\begin{matrix} \nu-\lambda+2, \nu-\lambda+\frac{3}{2} \\ \nu-\lambda-\frac{1}{2}n+\frac{3}{2} \end{matrix}; \frac{1}{2}-\frac{q}{2p}\right\} - \frac{1}{2}n \cdot {}_2F_1\left\{\begin{matrix} \nu-\lambda+\frac{3}{2}, \nu-\lambda+1 \\ \nu-\lambda-\frac{1}{2}n+\frac{3}{2} \end{matrix}; \frac{1}{2}-\frac{q}{2p}\right\}, \\ & \mathbf{R}(\nu-\lambda+1) > 0, \mathbf{R}(p) > p_0 > 0 \text{ and } |p| > |q|. \end{aligned}$$

11. Theorem VI. *If*

$$\varphi_{n,\lambda}(p) \stackrel{\frac{1}{2}n+\frac{1}{2}}{\pm\frac{1}{2}} x^{-\lambda} f(x)$$

and

$$\Phi'_{n,\lambda}(p) \stackrel{\frac{1}{2}n+\frac{1}{2}}{\pm\frac{1}{2}} x^{-\lambda} f(x),$$

then

$$\Phi'_{n-1,\lambda}(p) = p^{\frac{1}{2}} \varphi_{n-1,\lambda-\frac{1}{2}}(p) - 2^{\frac{1}{2}} \varphi_{n,\lambda}(p)$$

provided $\mathbf{R}(\mu - \lambda + 1) > 0$, where $f(x) = O(x^\mu)$ for small x and $\mathbf{R}(p) > p_0 > 0$, where $\Phi'_{n,\lambda}(p)$ is the image with respect to $D_n\{2(px)^{\frac{1}{2}}\}$.

Proof: We have

$$\varphi_{n,\lambda}(p) = 2^{-n/2} p \int_0^\infty D_n\{2(px)^{\frac{1}{2}}\} x^{-\lambda} f(x) dx \quad (1)$$

and

$$\Phi'_{n-1,\lambda}(p) = 2^{-\frac{1}{2}(n-1)} p \int_0^\infty D'_{n-1}\{2(px)^{\frac{1}{2}}\} x^{-\lambda} f(x) dx. \quad (2)$$

Using (Goldstein, 1932, p. 115)

$$D'_{n-1}(x) = \frac{1}{2} x D_{n-1}(x) - D_n(x),$$

(1) and (2), we get

$$\Phi'_{n-1,\lambda}(p) = p^{\frac{1}{2}} \varphi_{n-1,\lambda-\frac{1}{2}}(p) - 2^{\frac{1}{2}} \varphi_{n,\lambda}(p). \quad (3)$$

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NOTE ON A FUNCTIONAL EQUATION, CONNECTED WITH THE WEIERSTRASSIAN FUNCTION $\wp(z)$

By

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INTRODUCTION. This short paper, consisting only of two articles, aims at the *complete* solution of the functional equation:

$$\begin{vmatrix} f(x), & f(y), & f(x+y) \\ f'(x), & f'(y), & -f'(x+y) \\ 1, & 1, & 1 \end{vmatrix} = 0, \quad (x \neq y), \quad (I)$$

subject to the usual limitation that $f(z)$ shall be analytic *except for poles* in the *finite* part of Argand plane. An obvious implication of the well-known Addition Theorem is that a *particular* solution of (I) is

$$f(z) = \wp(z),$$

where $\wp(z)$ is the Weierstrassian elliptic function. The discovery of the *general* solution is, however, the main objective of the present investigation.

We are not aware whether the problem in the present form has been attempted heretofore by any previous writer.

1. At the very outset we remark that, for a *non-degenerate* function $f(z)$, satisfying (I), the origin must be a singularity. For, otherwise, when the point ($z = 0$) is supposed to be a regular point, we must have

$$f(0) = a \quad \text{and} \quad f'(0) = b$$

where a and b are *finite* constants. Then setting $y \doteq 0$ in (I) and allowing full variation to the other variable x , we get

$$\begin{vmatrix} f(x), & a, & f(x) \\ f'(x), & b, & -f'(x) \\ 1, & 1, & 1 \end{vmatrix} = 0.$$

This relation being equivalent to

$$f'(x)\{f(x) - a\} \doteq 0,$$

the legitimate inference is that $f(x)$ is a constant. Hence we can assert that *if a function $f(z)$, other than a constant, satisfies (I), it must have the point $z = 0$ for a singularity.* Further, this singularity must be *polar*, seeing that the essential singularities (in the *finite* part of the plane) are out of the picture in the present set-up.

We shall now suppose that the order of the pole (*vis.*, $z = 0$) is n , the associated principal part being

$$\frac{a_1}{z} + \frac{a_2}{z^2} + \dots + \frac{a_n}{z^n}, (a_n \neq 0). \quad (1)$$

Let us now revert to (I), and put

$$y = \epsilon, \quad (2)$$

where ϵ is a very small quantity (real or complex). Then by (1), we have

$$f(\epsilon) \sim \frac{a_1}{\epsilon} + \frac{a_2}{\epsilon^2} + \dots + \frac{a_n}{\epsilon^n}, \quad (3)$$

and

$$f'(\epsilon) \sim -\frac{a_1}{\epsilon^2} - \frac{2a_2}{\epsilon^3} - \dots - \frac{na_n}{\epsilon^{n+1}}. \quad (4)$$

The other variable x being, as before, supposed to be *unrestricted*, the equation (I) can be expanded in the form

$$f(x)[f'(\epsilon) + f'(x+\epsilon)] - f(\epsilon)[f'(x) + f'(x+\epsilon)] + f(x+\epsilon)[f'(x) - f'(\epsilon)] = 0. \quad (5)$$

The approximate values of $f(\epsilon)$ and $f'(\epsilon)$, as given by (3) and (4), being made use of, and Taylor's theorem being applied to $f(x+\epsilon)$ and $f'(x+\epsilon)$, the relation (5) becomes

$$\begin{aligned} f(x) \left[-\sum_{m=1}^{m=n} \frac{ma_m}{\epsilon^{m+1}} + \sum_{m=0}^{m=\infty} \frac{\epsilon^m}{m!} f^{(m+1)}(x) \right] - \left(\sum_{m=1}^{m=n} \frac{a_m}{\epsilon^m} \right) \times \left[f'(x) + \sum_{m=0}^{\infty} \frac{\epsilon^m}{m!} f^{(m+1)}(x) \right] \\ + \left\{ \sum_{m=0}^{\infty} \frac{\epsilon^m}{m!} f^{(m)}(x) \right\} \left[f'(x) + \sum_{m=1}^{m=n} \frac{ma_m}{\epsilon^{m+1}} \right] = 0. \end{aligned}$$

When multiplied by ϵ^{n+1} , the equation last written is carried over into

$$\begin{aligned} f(x) [-\{na_n + (n-1)a_{n-1}\epsilon + (n-2)a_{n-2}\epsilon^2 + \dots + 2a_2\epsilon^{n-2} + a_1\epsilon^{n-1}\} + \epsilon^{n+1}\{f''(x) + \epsilon f'''(x) + \dots\}] \\ - \epsilon(a_n + a_{n-1}\epsilon + a_{n-2}\epsilon^2 + \dots + a_1\epsilon^n)[2f'(x) + \epsilon f''(x) + \frac{\epsilon^2}{2!}f'''(x) + \dots] \\ + [f(x) + \epsilon f'(x) + \frac{\epsilon^2}{2!}f''(x) + \dots][na_n + (n-1)a_{n-1}\epsilon + \dots + a_1\epsilon^{n-1} + f'(x)\epsilon^{n+1}] = 0. \quad (6) \end{aligned}$$

As is obvious on inspection, the *finite* terms in the left side of (6) neutralise each other. Next, the least value of n being unity, it is manifest that the infinitesimal terms of the *first order* are together equal to

$$\epsilon(n-2)a_nf'(x). \quad (7)$$

Inasmuch as $a_n \neq 0$ and $f'(x) \neq 0$ (for a *non-degenerate* function), the identical vanishing of (7) implies

$$n-2=0,$$

showing that the pole (at $z=0$) must be of the *second order*. In other words, *if a non-degenerate function satisfies (I), it must have the origin for a pole of the second order*. The far-reaching consequences of this result will be traced fully in the succeeding article.

2. The origin ($z=0$) being proved to be a quadratic pole of the function $f(z)$, we may take its principal part at that point to be

$$\frac{a_1}{z} + \frac{a_2}{z^2}, \quad (a_2 \neq 0).$$

Without going over the whole ground afresh, we may simply write

$$a_3, a_4, \dots, a_n \text{ each} = 0,$$

so that the relation (6) of the preceding article at once boils down to

$$\begin{aligned} f(x) [- (2a_2 + a_1\epsilon) + \epsilon^3 \{ f'(x) + \epsilon f''(x) + \dots \}] - \epsilon (a_2 + a_1\epsilon) [2f'(x) + \epsilon f''(x) + \frac{\epsilon^2}{2!} f'''(x) + \dots] \\ + [f(x) + \epsilon f'(x) + \frac{\epsilon^2}{2!} f''(x) + \dots] [2a_2 + a_1\epsilon + \epsilon^3 f'(x)] = 0. \end{aligned} \quad (1)$$

As noticed already, the *finite* terms and the coefficient of ϵ in the left side of (1) vanish separately. Next the coefficient of ϵ^3 in the left side is easily seen to be $-a_1 f'(x)$. Since $f'(x) \neq 0$, (by hypothesis), we must have

$$a_1 = 0. \quad (2)$$

If we now equate to zero the coefficient of ϵ^3 in the left side of (1), and attend to the relation (2), we arrive at the relation

$$2f(x)f'(x) - \frac{1}{2}a_2 f''(x) = 0, \quad (3)$$

which can be treated as a differential equation to determine $f(x)$.

If we now write z for x , and introduce the notation $w = f(z)$, (3) assumes the form

$$w \frac{dw}{dz} = \frac{a_2}{12} \frac{d^3 w}{dz^3},$$

which can be readily integrated in the form

$$a_2 \left(\frac{dw}{dz} \right)^2 = 4w^3 - g_2 w - g_3, \quad (4)$$

where g_2 and g_3 are arbitrary constants.

The independent variable z being changed into $\lambda z'$, (where λ stands for the constant $\sqrt{a_2}$), (4) may be re-written as

$$\left(\frac{dw}{dz'} \right)^2 = 4w^3 - g_2 w - g_3. \quad (5)$$

Remembering that w , i.e., $f(z)$ becomes infinite as z (or z') $\rightarrow 0$, we can invert the relation (5) in the form

$$z' = \int_w^\infty \frac{dt}{(4t^3 - g_2 t - g_3)^{\frac{1}{2}}},$$

showing that

$$w = \mathfrak{E}(z') = \mathfrak{E}(\lambda z).$$

The sum and substance of the whole investigation may then be presented as follows:

The most general form of a non-degenerate function $f(z)$, which satisfies the functional equation

$$\begin{vmatrix} f(x), & f(y), & f(x+y) \\ f'(x), & f'(y), & -f'(x+y) \\ 1, & 1, & 1 \end{vmatrix} = 0, (x \neq y), \quad (I)$$

and has no essential singularities in the finite part of the plane, is that given by

$$f(z) = \mathfrak{E}(\lambda z),$$

it being understood that the parameter λ as well as the moduli g_2, g_3 , attaching to the elliptic function, are all arbitrary constants.

We shall now close this topic by simply mentioning that the proposition, established as above, indirectly provides an *independent* definition of the Weierstrassian elliptic function. In point of fact, it is easy to see that $\mathfrak{E}(z)$ is definable as the *uniquely determinate function, which satisfies the equation (I) and is analytic save as to poles in the finite part of the plane and besides harmonises with the special condition.*

$$\lim_{z \rightarrow 0} \{z^2 f(z)\} = 1$$

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A NOTE ON PARAMETER OF DISTRIBUTION OF A RULED SURFACE THROUGH A LINE OF A RECTILINEAR CONGRUENCE

By

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(Communicated by the Secretary—Received October 14, 1949—Revised on December 8, 1949)

The object of this note is to find out a new expression for the parameter of distribution of a ruled surface through a line of a rectilinear congruence. New expressions for developable and other surfaces of the congruence have also been obtained. Some properties have been deduced with the help of these expressions.

1. Let the coordinates of a point on the surface of reference be given by x^i ($i = 1, 2, 3$) and the direction cosines of the ray 'l' by λ^i ($i = 1, 2, 3$). Then the parameter of distribution of a ruled surface through 'l' is given by (Rām Behari, 1946)

$$d = \frac{(x^i_{,a} du^a - \lambda^i - \lambda^i_{,b} du^b)}{G_{a\beta} du^a du^\beta} \quad (1.1)$$

where $x^i_{,a}$ and $\lambda^i_{,a}$ are the covariant differentiations of x^i and λ^i and $G_{a\beta} \equiv \lambda^i_{,a} \cdot \lambda^i_{,\beta}$.

Squaring (1.1) we get,

$$d^2 = \frac{\begin{vmatrix} \bar{g}_{a\beta} du^a du^\beta & p_a du^a & \mu_{\beta a} du^a du^\beta \\ p_a du^a & 1 & 0 \\ \mu_{a\beta} du^a du^\beta & 0 & G_{a\beta} du^a du^\beta \end{vmatrix}}{G_{a\beta} G_{\gamma\delta} du^a du^\beta du^\gamma du^\delta} \quad (1.2)$$

where

$$\bar{g}_{a\beta} \equiv x^i_{,a} \cdot x^i_{,\beta}; \quad p_a \equiv \lambda^i_{,a} \cdot x^i_{,a} \quad \text{and} \quad \mu_{a\beta} \equiv \lambda^i_{,\beta} \cdot x^i_{,a}$$

or

$$d^2 = \frac{(\bar{g}_{a\beta} - p_a p_\beta) du^a du^\beta}{G_{a\beta} du^a du^\beta} - \left[\frac{\mu_{a\beta} du^a du^\beta}{G_{a\beta} du^a du^\beta} \right]^2$$

or

$$d^2 + t^2 = \frac{(\bar{g}_{a\beta} - p_a p_\beta) du^a du^\beta}{G_{a\beta} du^a du^\beta} \quad (1.3)$$

where t is the distance from the central point to the surface of reference.

For a developable surface

$$d = 0.$$

Therefore for a developable surface of the congruence the distance of the central point from the surface of reference is given by

$$t^2 = \frac{(\bar{g}_{a\beta} - p_a p_\beta) du^a du^\beta}{G_{a\beta} du^a du^\beta}. \quad (1.4)$$

Similarly the parameter of distribution of the surfaces for which the lines of striction lie on the surface of reference is given by

$$d^2 = \frac{(\bar{g}_{\alpha\beta} - p_\alpha p_\beta) du^\alpha du^\beta}{G_{\alpha\beta} du^\alpha du^\beta}. \quad (1.5)$$

2. The functions $x_{,\alpha}^i$ may be expressed in terms of λ^i and $\lambda_{,\alpha}^i$. Thus

$$x_{,\alpha}^i = p_\alpha \cdot \lambda^i + \mu_\alpha^\gamma \cdot \lambda_{,\gamma}^i, \quad (2.1)$$

where

$$\lambda_{,\beta}^i \cdot x_{,\alpha}^i = \mu_\alpha^\gamma \cdot \lambda_{,\gamma}^i \cdot \lambda_{,\beta}^i = \mu_\alpha^\gamma G_{\gamma\beta} \quad (2.2)$$

or

$$\mu_{\alpha\beta} G^{\gamma\beta} = \mu_\alpha^\gamma \quad (2.3)$$

which according to notation used by Eisenhart (1909) gives

$$\mu_1^i = \frac{G^e - Ff'}{\mathcal{K}^2}; \quad \mu_1^2 = \frac{\mathcal{E}f' - Fe}{\mathcal{K}^2}; \quad \mu_2^1 = \frac{Gf - Fg}{\mathcal{K}^2}; \quad \mu_2^2 = \frac{\mathcal{E}g - Fj}{\mathcal{K}^2}. \quad (2.4)$$

From (2.1) we get,

$$x_{,\alpha}^i \cdot x_{,\beta}^i = p_\alpha \cdot p_\beta + \mu_\alpha^\gamma \cdot \mu_\beta^\delta \cdot \lambda_{,\gamma}^i \cdot \lambda_{,\delta}^i$$

$$\bar{g}_{\alpha\beta} - p_\alpha \cdot p_\beta = \mu_\alpha^\gamma \cdot \mu_\beta^\delta \cdot G_{\gamma\delta} = \mu_\alpha^\gamma \cdot \mu_{\beta\gamma}. \quad (2.5)$$

Therefore the equation (1.8) assumes the form,

$$d^2 + t^2 = \frac{\mu_\alpha^\gamma \mu_{\beta\gamma} du^\alpha du^\beta}{G_{\alpha\beta} du^\alpha du^\beta}. \quad (2.6)$$

When expanded this equation becomes

$$d^2 + t^2 = \frac{(G^2 - 2Fef' + \mathcal{E}f'^2)(du^1)^2 + 2(Gef - Feg - Fff' + \mathcal{E}gf')du^1 du^2 + (Gf^2 - 2Fgf + \mathcal{E}g^2)(du^2)^2}{\mathcal{K}^2[\mathcal{E}(du^1)^2 + 2Fdu^1 du^2 + G(du^2)^2]} \quad (2.7)$$

Consequently for the developable surfaces,

$$t^2 = \frac{\mu_\alpha^\gamma \mu_{\beta\gamma} du^\alpha du^\beta}{G_{\alpha\beta} du^\alpha du^\beta}$$

$$t^2 = \frac{\mu_{\alpha\delta} \mu_{\beta\gamma} G^{\gamma\delta} du^\alpha du^\beta}{G_{\alpha\beta} du^\alpha du^\beta} \quad (2.8)$$

and for the surfaces whose lines of striction lie on the surface of reference

$$d^2 = \frac{\mu_\alpha^\gamma \mu_{\beta\gamma} du^\alpha du^\beta}{G_{\alpha\beta} du^\alpha du^\beta} \quad (2.9)$$

or

$$d^2 = \frac{\mu_{\alpha\delta} \mu_{\beta\gamma} G^{\gamma\delta} du^\alpha du^\beta}{G_{\alpha\beta} du^\alpha du^\beta}. \quad (2.10)$$

3. (a) In a congruence of Ribaucour (Eisenhart, 1909, p. 390)

$$\bar{g}_{\alpha\beta} - p_\alpha \cdot p_\beta = g_{\alpha\beta} \varphi^2$$

where $g_{\alpha\beta}$ are the fundamental tensor for the director surface and φ is the Bianchi's characteristic function.

Hence in a congruence of Ribaucour the equation (1.8) assumes the form

$$d^2 + t^2 = \frac{\varphi^2 g_{\alpha\beta} du^\alpha du^\beta}{G_{\alpha\beta} du^\alpha du^\beta} \quad (8.1)$$

and for the developable surfaces through a line of congruence of Ribaucour

$$t^2 = \frac{\varphi^2 g_{\alpha\beta} du^\alpha du^\beta}{G_{\alpha\beta} du^\alpha du^\beta}. \quad (8.2)$$

For a congruence of Ribaucour, the equation of surfaces of distribution is given by

$$t = 0.$$

Hence from (8.2) we get the result:

In a congruence of Ribaucour developable surfaces cut the surfaces of distribution along curves corresponding to null lines on the director surface.

If the congruence of Ribaucour is isotropic,

$$g_{\alpha\beta} = G_{\alpha\beta} \quad \text{and} \quad \mu_{\alpha\beta} + \mu_{\beta\alpha} = 0$$

and the equation (8.2) for developable surfaces becomes

$$\varphi^2 G_{\alpha\beta} du^\alpha du^\beta = 0$$

or

$$G_{\alpha\beta} du^\alpha du^\beta = 0. \quad (8.3)$$

Hence we get the well known result that *in an isotropic congruence the developable surfaces are represented on the sphere by minimal lines.*

(b) From the equation (1.8), $(d^2 + t^2)$ is positive. Therefore

$$(\bar{g}_{\alpha\beta} - p_\alpha p_\beta) du^\alpha du^\beta$$

is positive. Hence

$$\bar{g}_{\alpha\alpha} > (p_\alpha)^2, \quad i.e., \quad \bar{g}_{11} > p_1^2 \quad \text{and} \quad \bar{g}_{22} > p_2^2 \quad (8.4)$$

and

$$\begin{vmatrix} \bar{g}_{11} - p_1^2 & \bar{g}_{12} - p_1 p_2 \\ \bar{g}_{21} - p_2 p_1 & \bar{g}_{22} - p_2^2 \end{vmatrix} > 0 \quad (8.5)$$

which assumes the form

$$(\bar{g}_{11}\bar{g}_{22} - \bar{g}_{12}^2) - (\bar{g}_{11}p_2^2 + \bar{g}_{22}p_1^2 - 2\bar{g}_{12}p_1p_2) > 0$$

or

$$\frac{\bar{g}_{11}p_2^2 + \bar{g}_{22}p_1^2 - 2\bar{g}_{12}p_1p_2}{\bar{g}_{11}\bar{g}_{22} - \bar{g}_{12}^2} < 1.$$

The expression on the left denotes $\sin^2 \theta$ (Behari and Mishra, 1949), where θ is the

angle between a ray of the congruence and the normal to the surface of reference at the point where the ray intersects it.

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CORRECTIONS TO MY PAPER ON 'PARALLEL DISPLACEMENT AND SCALAR PRODUCT OF VECTORS—III'

By

R. N. SEN, Calcutta

The following errors have unfortunately crept into my paper entitled 'Parallel displacement and scalar product of vectors—III' published in *Bull. Cal. Math. Soc.*, 41, 113 (1949).

Page 114, formula (2.1), for $-T_{ia}^t$ read $+T_{ia}^t$.

Page 114, formula (2.2), for $+1/2$ read $-1/2$.

Page 116, 18th line from the top, for Δ_{ijk}^t read ∇_{ijk}^t .

Page 116, formula (3.2), for $\Delta_{ijk}^t - \frac{1}{2}(\bar{\Delta}_{ijk}^t + \bar{\Delta}_{ijk}'^t)$ read $2[\Delta_{ijk}^t - \frac{1}{2}(\bar{\Delta}_{ijk}^t + \bar{\Delta}_{ijk}'^t)]$.

Page 117, 12th, 18th lines from the top, equations on the left-side, for $\frac{1}{2}g^{tt}(g_{pt})_q$, $\frac{1}{2}g^{tt}(g_{qt})_p$ read $g^{tt}(g_{pt})_q$, $g^{tt}(g_{qt})_p$ respectively.

Page 117, equations (3.13), (3.14), for $1/16$ read $1/4$.

Page 117, last two lines, for $-8\{\alpha_{ijk}^t - \dots\} - 8\{\beta_{ijk}^t - \dots\}$ read $-2\{\alpha_{ijk}^t - \dots\} - 2\{\beta_{ijk}^t - \dots\}$.

Page 118, formula (3.17), for the terms $4K_{ijk}^t$, $-6\nabla_{ijk}^t$, $-3(\bar{\nabla}_{ijk}^t + \bar{\nabla}_{ijk}'^t)$,

$+8(\alpha_{ijk}^t + \beta_{ijk}^t)$, $+4(\bar{\Delta}_{ijk}^t + \bar{\Delta}_{ijk}'^t)$ read $2K_{ijk}^t$, $-\nabla_{ijk}^t$, $+\frac{1}{2}(\bar{\nabla} + \bar{\nabla}_{ijk}'^t)$,

$+2(\alpha_{ijk}^t + \beta_{ijk}^t)$, $+5(\bar{\Delta}_{ijk}^t + \bar{\Delta}_{ijk}'^t)$ respectively.

DEPARTMENT OF PURE MATHEMATICS,
CALCUTTA UNIVERSITY.

NOTE ON A TRIAD OF FUNCTIONAL EQUATIONS CONNECTED WITH THE LAGUERRE'S POLYNOMIAL $L_n(z)$

By

HARIDAS BAGCHI, *Calcutta*, and NALINI KANTA CHAKRABARTI, *Krishnagar*

(Received December 19, 1949—Revised on February 13, 1950)

INTRODUCTION. The main object of this short paper is to study the inter-relations, subsisting among the three functional equations

$$f'_n(z) = n[f'_{n-1}(z) - f_{n-1}(z)], \quad (I)$$

$$f_{n-1}(z) - (2n+1-z)f_n(z) + n^2 f'_{n-1}(z) = 0, \quad (II)$$

$$zf'_n(z) = n f_n(z) - n^2 f_{n-1}(z), \quad (III)$$

and the differential equation

$$z \frac{d^2 w}{dz^2} + (1-z) \frac{dw}{dz} + nw = 0, \quad (w = f_n(z)) \quad (A)$$

which are all associated with the Laguerre's polynomial (Courant and Hilbert, 1931; Sastry, 1934; Bagchi and Chakrabarti, 1950). Unless otherwise stated, the parameter n will be supposed to be *arbitrary*, and the equation (A) will be designated as the L -equation of rank n and symbolised simply as $L^{(n)}$. The gist of the results established in this paper is that the four equations (I), (II), (III) and (A) are tantamount to *two* effective equations. To that end it is necessary to reckon with a total of $4C_2$ (or 6) distinct *pairs* that can be formed out of the four equations. The six corresponding problems have been disposed of, three by three, in the first two articles. The third article is rather supplementary to the first two and practically gives a finishing touch to the whole investigation. At the end of the paper there is a passing reference to an *enumerably infinite* set of functions, satisfying any two of the four equations.

Finally we beg to express our indebtedness to our learned referee for his valuable criticisms and observations.

We are not aware whether the problem in the present form has been tackled heretofore by any previous writer.

1. In this article we propose to take account of three specific problems arising out of the three combinations of equations, *viz.*,

(i) I and II; (ii) I and III; and (iii) II and III.

To the next article we postpone the consideration of the remaining three problems, dealing respectively with the other three combinations:

(iv) I and A; (v) II and A; and (vi) III and A.

Problem (i) That the combination (I) and (II) leads automatically to (III) and (A) has been demonstrated fully in the authors' paper, cited already. So further discussion is uncalled for and hence dropped altogether.

Problem (ii). When the relations (I) and (III) are given in the first instance, we have, on changing n into $n+1$,

$$f'_{n+1}(z) = (n+1)[f'_n(z) - f_n(z)], \quad (1)$$

and

$$zf'_{n+1}(z) = (n+1)f'_{n+1}(z) - (n+1)^2 f_n(z). \quad (2)$$

Elimination of $f'_n(z)$ and $f'_{n+1}(z)$ from (1), (2) and (III) and subsequent simplification give rise to

$$f_{n+1}(z) - (2n+1-z)f_n(z) + n^2 f_{n-1}(z) = 0,$$

showing that (II) is a mere consequence of the combination (I) and (III). Thus Problem (ii) reduces to Problem (i), so that according to proved results, (A) will also follow as a matter of course.

Problem (iii). When (II) and (III) are given, we may alter n into $n-1$; so that

$$f_n(z) - (2n-1-z)f_{n-1}(z) + (n-1)^2 f_{n-2}(z) = 0 \quad (3)$$

and

$$-zf'_{n-1}(z) + (n-1)f'_{n-1}(z) - (n-1)^2 f_{n-2}(z) = 0 \quad (4)$$

Dispensing with $f_n(z)$ and $f_{n-2}(z)$ from (3), (4) and (III), we find after easy reductions

$$f'_n(z) = n[f'_{n-1}(z) - f_{n-1}(z)].$$

This being the same as (I), we infer, on the strength of the results of Problems (i) and (ii) that (II) and (III), taken together, affirm both (I) and (A).

The remaining three problems will be disposed of in the next article.

2. Problem (iv). Starting with the combination (I) and (A), we have, on taking two values (say, n and $n+1$) of the parameter,

$$f'_n(z) = nf'_{n-1}(z) - nf_{n-1}(z), \quad (1)$$

$$zf'_n(z) + (1-z)f'_n(z) + nf_n(z) = 0, \quad (2)$$

as well as

$$f'_{n+1}(z) = (n+1)f'_n(z) - (n+1)f_n(z) \quad (3)$$

and

$$zf'_{n+1}(z) + (1-z)f'_{n+1}(z) + (n+1)f_{n+1}(z) = 0. \quad (4)$$

Evidently (3) gives on differentiation

$$f'_{n+1}(z) = (n+1)f'_n(z) - (n+1)f'_n(z). \quad (5)$$

Eliminating $f'_n(z)$, $f'_n(z)$ and $f'_{n+1}(z)$ from (2), (3), (4) and (5), we eventually arrive at the relation

$$zf'_{n+1}(z) = (n+1)f_{n+1}(z) - (n+1)^2 f_n(z).$$

This being substantially the same as (III), the inevitable conclusion is that (III) is but a corollary to the combination (I) and (A). Consequently, by Problem (ii) we conclude further that the same combination leads also to (II).

Problem (v). When (II) and (A) are given in the first instance, we may choose three special values n , $n+1$ and $n-1$ for the parameter in (A), so that we have at the very start

and
$$zf_r'(z) + (1-z)f_r'(z) + rf_r(z) = 0, \quad (r = n+1, n, n-1), \quad (6), (7), (8)$$

$$f_{n+1}(z) - (2n+1-z)f_n(z) + n^2f_{n-1}(z) = 0. \quad (II)$$

Two-fold differentiation of (II) gives

$$f_{n+1}'(z) - (2n+1-z)f_n'(z) + f_n(z) + n^2f_{n-1}'(z) = 0 \quad (9)$$

and

$$f_{n+1}'(z) - (2n+1-z)f_n'(z) + 2f_n'(z) + n^2f_{n-1}'(z) = 0. \quad (10)$$

If we now add (10), (multiplied by z), to (9), (multiplied by $1-z$), and then to (II), (multiplied by $n+1$), and further attend to (6), (7) and (8), we derive after easy reductions:

$$zf_n'(z) = nf_n(z) - n^2f_{n-1}(z).$$

This being the same as (III), we infer by Problem (ii) that the combination (II) and (A) implies (III) as well as (I).

Problem (vi). When we start with (III) and (A), we may choose values n and $n+1$ for the parameter in (III), so that we have initially

$$zf_n'(z) = nf_n(z) - n^2f_{n-1}(z), \quad (III)$$

$$zf_n'(z) = -\{(1-z)f_n'(z) + nf_n(z)\} \quad (A)$$

as well as

$$zf_{n+1}'(z) = (n+1)f_{n+1}(z) - (n+1)^2f_n(z). \quad (11)$$

Now differentiating (III) and coupling the derived relation with (A), we get

$$(z-n)f_n'(z) - nf_n(z) + n^2f_{n-1}(z) = 0.$$

$n+1$ being put for n , this becomes

$$(z-n-1)f_{n+1}'(z) - (n+1)f_{n+1}(z) + (n+1)^2f_n'(z) = 0. \quad (12)$$

Elimination of $f_n'(z)$ and $f_{n+1}'(z)$ from (III), (11) and (12), followed by elementary manipulations, gives

$$f_{n+1}(z) - (2n+1-z)f_n(z) + n^2f_{n-1}(z) = 0.$$

This relation being none other than (II), the logical conclusion is that (III) and (A), taken together, imply (II). By Art 1 (Problem ii), we gather that (I) will also follow from the combination of (III) and (A).

Thus (III) and (A) are together equivalent to (II) and (I).

3. The results of the six problems, dealt with separately in Arts. 1 and 2, may now be amalgamated and condensed in the following form:

The three functional equations, viz.,

$$f_n'(z) = n[f_{n-1}'(z) - f_{n-1}(z)], \quad (I)$$

$$f_{n+1}(z) - (2n+1-z)f_n(z) + n^2f_{n-1}(z) = 0, \quad (II)$$

$$zf_n'(z) = nf_n(z) - n^2f_{n-1}(z) \quad (III)$$

and the differential equation

$$z \frac{d^2 w}{dz^2} + (1-z) \frac{dw}{dz} + nw = 0, \quad [w \equiv f_n(z)] \quad (\text{A})$$

are so related to one another that if a sequence of functions $\{f_n(z)\}$ satisfies any two of them, it must satisfy the other two. In other words, the four equations are tantamount to two independent equations.

It is clear that, in so far as this general proposition is concerned, the parameter n may be perfectly arbitrary (real or complex). An interesting special case arises when n is restricted to be a positive integer, so that the sequence $\{f_n(z)\}$ is *enumerably infinite* and is composed of functions like

$$f_0(z), f_1(z), f_2(z), \dots, f_n(z), \dots \quad (\text{B})$$

Reference may now be made to the authors' paper (1950), where a synthetic method has been devised for finding an *enumerable* set of functions like (B), which satisfy both the functional equations (I) and (II) and therefore necessarily the differential equation (A).

In view of the general proposition, proved as above, it is abundantly clear that the self-same set of *enumerable* functions of the type (B), just talked about, might as well be regarded as representing the *common solutions* of *any two* of the four equations (I), (II), (III) and (A).

DEPARTMENT OF PURE MATHEMATICS,
CALCUTTA UNIVERSITY,
CALCUTTA

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CALCUTTA MATHEMATICAL SOCIETY

Report of the Council for the year 1949 to the Annual General Meeting of the Society

The Council of the Calcutta Mathematical Society has the pleasure to submit the following report on the general concerns of the Society for the year 1949 as required by the provisions of Rule 25.

The Council: The Council of the Society for the year 1949 consisting of the officers and members elected at the last Annual General Meeting together with the Editorial Secretary, was constituted as follows:

President

Prof. A. C. Banerjee

Vice-Presidents

Prof. F. W. Levi Prof. N. M. Basu

Dr. R. C. Bose Prof. B. B. Sen

Prof. V. V. Narlikar

Treasurer

Mr. S. C. Ghosh

Secretary

Mr. U. R. Burman

Editorial Secretary

Mr. P. K. Ghosh

Other Members of the Council

Dr. S. S. Pillai Dr. B. S. Ray

Dr. R. N. Sen Prof. N. R. Sen

Dr. T. Vijayaraghavan Dr. S. Ghosh

Mr. S. Gupta Prof. M. R. Siddiqi

Dr. B. R. Seth Prof. C. N. Srinivasiengar

Prof. P. N. Dasgupta Mr. B. N. Mukherjee

General: The various activities of the Society have been conducted throughout the year in accordance with the usual procedure. The council is glad to report that it was possible to arrange a lecture by Prof. S. Chapman, Sedlian Professor of Natural Philosophy in the University of Cambridge (England) during the Professor's visit to this country to attend the 36th session of the Indian Science Congress held at Allahabad

in January 1949. The Council offers its grateful thanks to Prof. Chapman for the illuminating lecture he delivered on "*Atmospheric oscillations and Lunar tides*".

Membership: The Council reports with satisfaction that there has been a number of additions to the list of ordinary members of the society, no fewer than 10 new members being elected during the year under review. The council still feels that every effort should be made to bring into the society as many as possible of those persons who would be glad to benefit by its activities once having been made properly aware of them.

Meetings During 1949: The Council held four meetings during the year and there were five ordinary meetings devoted to the reading of original papers communicated to the society for publication in the Bulletin.

Publications: During the year 1949, the society has published five numbers of the Bulletin, namely Vol 40, No. 4 and Vol. 41, Nos. 1-4. The council notes with satisfaction that the printing delays which have so seriously interfered with the regular appearance of the Bulletin have been overcome this year through the efforts of Mr. S. Gupta, a member of the present council and a former Secretary of the Society's Editorial Board. The council conveys its most sincere thanks to him. It is also necessary to record the Society's deep indebtedness to the authorities of the Calcutta University for printing the Bulletin free of charge and to the officers and members of the staff of the University Press who have given their every sympathetic and active co-operation in bringing out the Bulletin in time inspite of conditions in the printing industry being still very difficult.

Exchange of Publications: The transmission of the Society's publications to various countries in the world have been carried on in the usual manner and some new exchange relations have also been established during the year under review. The distribution of wartime issues of the Bulletin to all institutions on the exchange list has almost been completed and a few sets of earlier issues are yet being sent out to institutions whose libraries have been destroyed or badly damaged during the war. The wartime gaps in the Society's copies of foreign publications have mostly been filled up and attempts are being made to make our files as complete as possible in respect of all the available materials.

Library: The Council has the pleasure to report that the usefulness of the Library has been considerably enhanced by the addition of the following periodicals to the existing list:

Acta Mathematica, Mathematische Annalen, Quarterly Journal of Mathematics (Oxford series), Monthly Notices of the Royal Astronomical Society (Geophysical supplement), Communications of the Institute for Advanced Study, Dublin (Eire). Some more journals ordered for this year, are also expected to arrive early in 1950. The purchase of these journals has been made possible by a capital grant to the society made by the Government of India in pursuance of their policy of stimulating the spirit of scientific research in the country. The Council will speak about the details of this grant in this report under the Finance head. The council hopes that if the flow of such

periodicals into the society's Library can be maintained uninterrupted through the munificence of the Government of India, the society will be able to make very substantial contributions to the cause of Mathematical study and research in the country.

The demands on the shelf-space of the library has increased so much during the present year that the Council has decided to meet this requirement by the construction of additional steel shelves. This construction is likely to be completed in February, 1950 at an estimated cost of about Rs 1250/- and it is hoped that the increase in space will provide relief from congestion for a period of seven or eight years at the present rate of growth of the library.

Finance: The annual accounts of the society for the year 1949 have been presented to the Council in the standardized form by the auditors Dr. S. K. Chakrabarty and Dr. S. K. Basu. The Council gratefully acknowledges its indebtedness to them for their honorary services. The society received a sum of Rs. 500/- from the Government of India through the National Institute of Sciences, as grant-in-aid of publications and also a capital grant of Rs. 4000/- to be utilised for the development of the library and for the purchase of mathematical types and symbols. The former has been wholly used up in purchasing paper for the Bulletin and the additional journals to the library previously referred to in this report, have been paid for out of the latter grant. Orders for sets of more periodicals, to the extent of the amount still available for the purpose have already been placed with foreign subscription agencies, and the whole consignment is likely to be in hand by March 1950. The Council would take this opportunity of extending the society's most grateful thanks to the Government of India and the National Institute of Sciences for these grants.

The council also desires to mention that with a view to meet the increasing demands on the society's revenues it has been necessary to enhance the annual subscription price of the Bulletin with effect from 1950. The revised subscription price has therefore been fixed as follows:

India and Pakistan—Rs. 12/- (postage inclusive)

Elsewhere —Rs. 13/- „ „

This is estimated to bring an additional annual return of about Rs. 250/- on the revenue side. The council hopes to receive the same support from subscription agencies as it has hitherto done.

Changes in Rules & Regulations: The council after careful consideration proposed two amendments to the existing Rules and Regulations of the society and in accordance with the rules, put them in circulation amongst the members of the society to ascertain their views. The amendments refer to Sec. 2 and Sec. 40 of the standing Rules and Regulations; one gives power to the Council for co-option in its body not more than three persons from amongst the Benefactors while the other is concerned with the definition of Benefactors. The Council is pleased to report that its recommendations have been agreed to by the members and these will therefore be declared adopted at the Society's Annual General Meeting to-day.

RECEIPTS AND DISBURSEMENTS ACCOUNTS OF THE CALCUTTA MATHEMATICAL SOCIETY FOR THE YEAR ENDING 31ST DECEMBER, 1949.

The MEMBERS of the Calcutta Mathematical Society,
We have examined the above Balance Sheet with the B
accordance with the information and explanations given to us.

ON COMPLEX TENSOR CALCULUS

By

N. N. GHOSH, *Calcutta*

(Received January 31, 1960)

Introduction. The object of this paper is to give a new systematic development of the complex tensor calculus in close analogy to the ordinary treatment (Eddington, 1924), as restricted to real coordinates. It differs from the existing theory (Schouten-Struik, 1938), as being based on more general notions. Starting with the most general definition of a function of several complex variables and a general coordinate transformation, the procedure adopted evolves an extended summation convention, in consequence of which the notations and formulae can be presented in familiar forms, possessing, however, a wider significance. In the present theory the invariants are always real, while the vectors and the tensors have complex components obeying characteristic transformation laws.

Although the metric tensor is complex, the metric of the complex space is real and obviously a generalized form of the line-element defined in general relativity. As a result, Einstein's gravitational equations are generalized to admit of complex coordinates. As regards Maxwell's electromagnetic equations, a special metric is defined, which preserves its invariant form under Lorentz transformations. The theory ultimately leads to the generalized electromagnetic equations.

1. *Coordinates:* As an initial step to generalization, there will be n coordinates x^1, x^2, \dots, x^n , denoted briefly as x^μ , admitting complex values given by

$$x^\mu = q^\mu + ip^\mu, \quad i = (-1)^{\frac{1}{2}}, \quad (1.1)$$

where the quantities q^μ, p^μ are real. The system of coordinates, complex conjugate to (1.1), will be denoted by

$$x^{\mu*} = q^\mu - ip^\mu. \quad (1.2)$$

We shall call μ and μ^* a pair of *complementary* indices. Assuming the possibility of associating coordinates x^μ with the points of a complex space the totality of points determined by assigning arbitrary values to x^μ , i.e., to the $2n$ real and independent variables q^μ and p^μ , will constitute a complex space of n dimensions. A domain (D) of this complex space is defined by the set of values of x^1, x^2, \dots, x^n for which

$$|x^\mu| \leq r^\mu, \quad (\mu = 1, 2, \dots, n).$$

2. *Function of several complex variables:* A complex variable $M = m + il$ is said to be a function of x^μ for values belonging to a domain (D), if there exists an assignment of values of M to the values of x^μ in (D) such that, to every value of x^μ in (D) there corresponds a definite value of M , i.e., of m and l . In accordance with this general definition we, however, assume that m, l are single-valued continuous functions of $2n$ variables (q^μ, p^μ), together with their first and second derivatives.

It is easily verified that the coefficients of transformation in (5.2, 3) satisfy relations of the type

$$\left. \begin{aligned} (\xi_\alpha x'^\mu)(\xi_\mu^* x^\beta) &= \xi_\alpha x^\beta = 0, & (\xi_\alpha x'^\mu)(\xi_\mu^* x^{\beta*}) &= \xi_\alpha x^{\beta*} = 1, \text{ if } \alpha = \beta \\ & & &= 0, \text{ if } \alpha \neq \beta \end{aligned} \right\} \quad (5.4)$$

6. *Tensors of second and higher orders*: Considering the transformation laws of the product of a pair of contravariant vectors we are led to define a contravariant tensor $A^{\mu\nu}$ of second order having $4n^2$ components ($A^{\alpha\beta}, A^{\alpha\beta*}, A^{*\alpha\beta}, A^{*\alpha\beta*}$), where the third and the fourth denote respectively the complex conjugates of the second and the first. The $4n^2$ equations of transformation are then included in the formula

$$A'^{\mu\nu} = A^{\alpha\beta}(\xi_\alpha x'^\mu)(\xi_\beta^* x'^\nu), \quad (6.1)$$

where μ, ν range over values $1, 2, \dots, n, 1^*, 2^*, \dots, n^*$.

In the same way, the transformation laws with regard to a covariant tensor $A_{\mu\nu}$ and a mixed tensor A_μ^ν are respectively

$$A'_{\mu\nu} = A_{\alpha\beta}(\xi_\mu^* x'^\alpha)(\xi_\nu^* x'^\beta), \quad (6.2)$$

$$A'_\mu^\nu = A_\alpha^\beta(\xi_\mu^* x'^\alpha)(\xi_\beta^* x'^\nu), \quad (6.3)$$

where components with complementary indices are complex conjugates of one another.

The transformation laws for the tensors of higher order follow from the above by obvious generalizations. Thus a mixed tensor of order four $A_{\mu\nu}^{\rho\sigma}$, having $(2n)^4$ components will obey the transformation law

$$A'^{\rho\sigma}_{\mu\nu} = A^{\alpha\beta\gamma\delta}_{\mu\nu}(\xi_\mu^* x'^\alpha)(\xi_\nu^* x'^\beta)(\xi_\gamma x'^\rho)(\xi_\delta x'^\sigma), \quad (6.4)$$

where ρ, μ, ν, σ range over values $1, 2, \dots, n, 1^*, 2^*, \dots, n^*$ and components with complementary indices are complex conjugates of one another.

7. *Inner multiplication and contraction*: Given a contravariant vector A^μ and a covariant vector B_μ , the sum of $2n$ terms denoted by $A^{\mu*} B_\mu$ is called the inner product of the two vectors. Evidently this is real and making use of the transformation equations (5.2, 3) and the relation (5.4), it can be proved also to be an invariant. Similarly the inner product of a covariant tensor and a contravariant vector may be proved to be a covariant vector.

To illustrate the contraction of a tensor, let us consider the mixed tensor (6.4). Setting $\rho = \sigma^*$, we get

$$A'^{\sigma*}_{\mu\nu} = A^{\alpha\beta\gamma\delta}_{\mu\nu}(\xi_\mu^* x'^\alpha)(\xi_\nu^* x'^\beta)(\xi_\gamma x'^{\sigma*})(\xi_\delta x'^\sigma) = A^{\alpha\beta\gamma}_{\mu\nu}(\xi_\mu^* x'^\alpha)(\xi_\nu^* x'^\beta)(\xi_\gamma x'^{\sigma*}), \quad (7.1)$$

in consequence of (5.4). Hence $A^{\sigma*}_{\mu\nu}$ is a covariant tensor obeying the transformation law (6.2).

8. *The metric tensor*: Let $g_{\mu\nu}$ be a covariant tensor and dx^μ an infinitesimal vector, then the inner product $g_{\mu\nu} dx^\mu dx^{\nu*}$ is real and invariant. We further remark that the components

$$g_{\mu\nu} = g_{\nu\mu}, \quad g_{\mu\nu}^* = g_{\nu\mu} = \text{conj. } g_{\nu\mu^*}, \quad (8.1)$$

and call $g_{\mu\nu}$ the covariant metric tensor. As a scalar measure of the linear element ds corresponding to the infinitesimal vector dx^μ we shall take the equation

$$ds^2 = g_{\mu\nu} dx^{\mu*} dx^{\nu*}. \quad (8.2)$$

We now form the determinant of order $2n$

$$g = \begin{vmatrix} g_{1*1} & \dots & g_{1*n} & g_{1*1*} & \dots & g_{1*n*} \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ g_{n*1} & \dots & g_{n*n} & g_{n*1*} & \dots & g_{n*n*} \\ g_{11} & \dots & g_{1n} & g_{11*} & \dots & g_{1n*} \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ g_{n1} & \dots & g_{nn} & g_{n1*} & \dots & g_{nn*} \end{vmatrix} \quad (8.3)$$

which is hermitian, because of the relations (8.1). Let $g^{\mu\nu}$ and $g^{\mu\nu*}$ be defined as the cofactors of $g_{\nu*\mu}$ and $g_{\nu\mu}$ respectively in the determinant, divided by g ; then from well-known law of determinants, we have

$$\left. \begin{aligned} g_{\mu*}\sigma g^{\sigma*}{}_\nu &= g_{\mu*}^{\nu*} = 1, \text{ if } \mu = \nu \\ &= 0, \text{ if } \mu \neq \nu \end{aligned} \right\}, \quad g_{\mu*}\sigma g^{\sigma*}{}_{\nu*} = 0. \quad (8.4)$$

The components

$$g^{\mu\nu} = g^{\nu\mu}, \quad g^{\mu\nu*} = g^{\nu*\mu} = \text{conj. } g^{\mu\nu},$$

together with their complex conjugates will define the contravariant metric tensor $g^{\mu\nu}$.

9. Raising and lowering of indices of tensors: Raising the index of a covariant vector will be defined by the equation

$$A^{\mu*} = g^{\mu\nu*} A_\nu = g^{\nu\mu} A_{\nu*}, \quad (9.1)$$

and lowering the index of a contravariant vector by the equation

$$A_\mu = g_{\mu\nu} A^{\nu*} = g_{\nu\mu} A^{\nu*}. \quad (9.2)$$

Similarly we can raise or lower one or more indices in a given tensor. Such tensors are called associated tensors.

10. Christoffel symbols: The Christoffel symbol $[\mu\nu, \sigma]$ of the first kind will be defined by

$$[\mu\nu, \sigma] = \frac{1}{2}(\xi_\nu g_{\mu\sigma} + \xi_\mu g_{\nu\sigma} - \xi_\sigma g_{\mu\nu}). \quad (10.1)$$

Since μ, ν, σ range over values $1, 2, \dots, n, 1^*, 2^*, \dots, n^*$, there are four distinct types of such symbols, symbols with complementary indices being complex conjugates of one another. Christoffel symbols of the second kind are related to the first by means of the equation

$$\left\{ \begin{matrix} \sigma \\ \mu\nu \end{matrix} \right\} = g^{\lambda*\sigma} [\mu\nu, \lambda]. \quad (10.2)$$

It may be proved that

$$\left\{ \begin{matrix} \nu^* \\ \mu\nu \end{matrix} \right\} = \frac{1}{2g} \xi_\mu g = \xi_\mu \log \sqrt{g}, \quad (10.3)$$

assuming g to be positive.

$$A_{\mu\nu\sigma} = \xi_\sigma A_{\mu\nu} - \left\{ \begin{matrix} \alpha \\ \mu\sigma \end{matrix} \right\} A_{\alpha\nu} - \left\{ \begin{matrix} \alpha \\ \nu\sigma \end{matrix} \right\} A_{\mu\alpha}, \quad (12.3)$$

$$A_{\mu\nu}^\nu = \xi_\sigma A_{\mu}^\nu - \left\{ \begin{matrix} \alpha \\ \mu\sigma \end{matrix} \right\} A_{\alpha}^\nu + \left\{ \begin{matrix} \nu \\ \alpha\sigma \end{matrix} \right\} A_{\mu}^{\alpha*}, \quad (12.4)$$

$$A_{\mu\nu,\sigma}^{\mu\nu} = \xi_\sigma A^{\mu\nu} + \left\{ \begin{matrix} \mu \\ \alpha\sigma \end{matrix} \right\} A_{\alpha}^{\mu*} + \left\{ \begin{matrix} \nu \\ \sigma\sigma \end{matrix} \right\} A^{\mu\alpha*}. \quad (12.5)$$

A geodesic is defined by the equations expressed as

$$\frac{d^2 x^\alpha}{ds^2} + \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0. \quad (12.6)$$

Riemann-Christoffel tensor is defined by the equation

$$A_{\mu,\nu\sigma} - A_{\mu,\sigma\nu} = A_{\rho}{}^* R_{\mu\nu\sigma}^\rho,$$

where

$$R_{\mu\nu\sigma}^\rho = \left\{ \begin{matrix} \alpha \\ \mu\sigma \end{matrix} \right\} \left\{ \begin{matrix} \epsilon \\ \alpha^*\nu \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} \left\{ \begin{matrix} \epsilon \\ \alpha^*\sigma \end{matrix} \right\} + \xi_\nu \left\{ \begin{matrix} \epsilon \\ \mu\sigma \end{matrix} \right\} - \xi_\sigma \left\{ \begin{matrix} \epsilon \\ \mu\nu \end{matrix} \right\}. \quad (12.7)$$

By setting $\epsilon = \sigma^*$ and utilizing (10.2), we derive the contracted tensor

$$G_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\sigma \end{matrix} \right\} \left\{ \begin{matrix} \sigma^* \\ \alpha^*\nu \end{matrix} \right\} - \left\{ \begin{matrix} \alpha^* \\ \mu\nu \end{matrix} \right\} \xi_\alpha \log \sqrt{g} + \xi_\mu \xi_\nu \log \sqrt{g} - \xi_\sigma \left\{ \begin{matrix} \sigma^* \\ \mu\nu \end{matrix} \right\}, \quad (12.8)$$

which is the complex analogue of $G_{\mu\nu}$ in the theory of relativity. All such formulae are, however, to be interpreted in accordance with general principles underlying the present theory.

13. Electromagnetic metric tensor: We conclude this paper with the discussion of a special type of the metric (8.2) in which the components of the metric tensor are as follows :

$$g_{\mu\nu} = 0, \quad \text{Real part of } g_{\mu\nu}^* = 0, \quad (13.1)$$

μ, ν ranging over the values 1, 2, 3, 4.

Assume $g_{\mu\nu}^* = \frac{1}{2}if_{\mu\nu}$, where $f_{\mu\nu}$ is real, then since $g_{\mu\nu} = \text{conj. } g_{\nu\mu}^*$, we must have $f_{\mu\nu} = -f_{\nu\mu}$. We shall call $g_{\mu\nu}^*$ the *electromagnetic metric tensor*. To define $g^{\mu\nu}$, instead of proceeding with the determinant g in (8.3), it is simpler to take the determinant

$$a = |g_{\mu\nu}^*| \quad (13.2)$$

of order four, in which

$$g^{\mu\nu} = \frac{\text{cofactor of } g_{\nu\mu}^*}{a}.$$

Since $g_{\mu\nu}^*$ is purely imaginary, the diagonal elements of (13.2) vanish and we obtain

$$g^{21*} = -\frac{g_{34}^*}{(a)^{\frac{1}{4}}}, \quad g^{31*} = \frac{g_{24}^*}{(a)^{\frac{1}{4}}}, \quad g^{41*} = -\frac{g_{23}^*}{(a)^{\frac{1}{4}}}, \quad (13.3)$$

$$g^{32*} = -\frac{g_{14}^*}{(a)^{\frac{1}{4}}}, \quad g^{42*} = \frac{g_{13}^*}{(a)^{\frac{1}{4}}}, \quad g^{43*} = -\frac{g_{12}^*}{(a)^{\frac{1}{4}}}.$$

Consider now a Lorentz transformation expressed as

$$q'^1 = \frac{q^1 - uq^4}{(1-u^2)^{\frac{1}{2}}}, \quad q'^2 = q^2, \quad q'^3 = q^3, \quad q'^4 = \frac{q^4 - uq^1}{(1-u^2)^{\frac{1}{2}}}, \quad (13.4)$$

$$p'^1 = \frac{p^1 - up^4}{(1-u^2)^{\frac{1}{2}}}, \quad p'^2 = p^2, \quad p'^3 = p^3, \quad p'^4 = \frac{p^4 - up^1}{(1-u^2)^{\frac{1}{2}}}.$$

This gives

$$\xi'_\mu x^\mu = 0, \quad \xi_\mu x'^\mu = 0, \quad \xi'_\mu x^{\mu*} = \frac{\partial q^\alpha}{\partial q'^\mu}, \quad \xi_\mu x'^{\mu*} = \frac{\partial q'^\mu}{\partial q^\alpha}. \quad (13.5)$$

The equations of transformation (6.1, 2) are therefore much simplified and become

$$g'^{\mu\nu} = 0, \quad g'^{\mu\nu*} = g^{\alpha\beta*} \frac{\partial q'^\mu}{\partial q^\alpha} \cdot \frac{\partial q'^\nu}{\partial q^\beta}, \quad g'_{\mu\nu} = 0, \quad g'_{\mu\nu*} = g_{\alpha\beta*} \frac{\partial q^\alpha}{\partial q'^\mu} \cdot \frac{\partial q^\beta}{\partial q'^\nu}, \quad (13.6)$$

where the dummy indices run over values 1, 2, 3, 4 only. Hence the metric

$$ds^2 = g_{\mu\nu*} dx^\mu dx^\nu + g_{\mu\nu} dx^{\mu*} dx^{\nu*} \quad (13.7)$$

will preserve its form under a Lorentz transformation. It may be noted that the above reduces to the more explicit real form

$$ds^2 = 2 f_{\mu\nu} dp^\mu \cdot dq^\nu. \quad (13.8)$$

14. Maxwell equations: We proceed next to obtain the first set of Maxwell equations. Since $\xi'_\lambda = \frac{\partial q^\gamma}{\partial q'^\lambda} \xi_\gamma$; we have from (13.6)

$$\xi'_\lambda g'_{\mu\nu*} = \xi_\gamma g_{\alpha\beta*} \frac{\partial q^\alpha}{\partial q'^\mu} \cdot \frac{\partial q^\beta}{\partial q'^\nu} \cdot \frac{\partial q^\gamma}{\partial q'^\lambda} \quad (14.1)$$

Similarly,

$$\xi'_\mu g'_{\nu\lambda*} = \xi_\alpha g_{\beta\gamma*} \frac{\partial q^\alpha}{\partial q'^\mu} \cdot \frac{\partial q^\beta}{\partial q'^\nu} \cdot \frac{\partial q^\gamma}{\partial q'^\lambda} \quad (14.2)$$

$$\xi'_\nu g'_{\lambda\mu*} = \xi_\beta g_{\gamma\alpha*} \frac{\partial q^\alpha}{\partial q'^\mu} \cdot \frac{\partial q^\beta}{\partial q'^\nu} \cdot \frac{\partial q^\gamma}{\partial q'^\lambda} \quad (14.3)$$

Adding (14.1, 2, 3) together it appears that under a Lorentz transformation

$$\xi'_\lambda g_{\mu\nu*} + \xi'_\mu g_{\nu\lambda*} + \xi'_\nu g_{\lambda\mu*} \quad (14.4)$$

is a tensor of the third order. Hence the first set of Maxwell equations follows.

Let us introduce a covariant vector A_μ , the electromagnetic potential, and set

$$g_{\mu\nu*} = i(\xi_\mu A_\nu - \xi_\nu A_\mu), \quad (14.5)$$

then obviously (14.4) vanishes. It should be remembered that A_μ must be such that $\xi_\mu A_\nu - \xi_\nu A_\mu$ is real.

To obtain the second set of Maxwell equations we introduce the elementary tensor having components

$$A^{11} = A^{22} = A^{33} = 1, \quad A^{44} = -1, \quad A^{\mu\nu} = 0, \quad A^{\mu\nu*} = 0. \quad (14.6)$$

Let us form the contravariant tensor

$$A^{\mu\nu}A^{\lambda\delta*}g_{\lambda\nu*} = h^{\mu\delta*} \quad (14.7)$$

then the second set of Maxwell equations may be presented in the form

$$\xi_\delta h^{\mu\delta*} = J^\mu, \quad (14.8)$$

where J_μ is the charge-and-current vector.

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NOTE ON A CIRCULAR CUBIC WITH A REAL INFLEXION AT INFINITY

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INTRODUCTION. As is well-known, the subject of circular cubics is a classical one, its "circles of inversion" and "focal parabolas" being associated with Prof. Casey; latterly the subject attracted the attention of a band of prominent mathematicians, the majority of whom studied the cubic as a *degenerate* variety of a bicircular quartic. The present investigation centres round that *particular* variety of circular cubic, which has a real inflexion at infinity. Although we have confined our attention chiefly to *bicursal*† circular cubics, still there are occasional references to *unicursal* (circular) cubics (nodal and cuspidal).

As a matter of convenience, the subject has been subdivided into four sections. In the first place Sec I deals with the *geometrical* aspect of the subject, its method being based on Sylvester's Theory of Residuation. Then the next two sections (II) and (III) take account of the *analytic* aspect, and make free use of rectangular Cartesian coordinates to the exclusion of trilinear or other types of homogeneous coordinates, this being necessitated by the fact of the cubic having to pass through the two *circular* points at infinity. It is remarkable that although the discussions of both Sections II and III are based on Cartesian methods, still there is an essential difference in their mode of discussion. Thus while on the one hand Sec. II reckons with the Cartesian equations of the four "circles of inversion" and the four associated "focal parabolas" of an unrestricted circular cubic (given in the Cartesian form) with a laconic reference to certain special varieties, *e.g.*, a Trisectrix of Maclaurin, Sec. III, on the other hand, utilises special canonical forms of a bicursal circular cubic, having one or more inflexions at infinity, to demonstrate some of its more salient properties. Finally, Sec. IV concerns itself principally with unicursal circular cubics, having one or more inflexions at infinity.

Although on certain occasions it has been felt necessary to touch on *known* results, still this paper is believed to embody a considerable amount of original matter.

SECTION I

Geometrical treatment based on the Theory of Residuation

1. Suppose that Γ is a bicursal circular cubic, having its real point (K) at infinity for an inflexion and that α is the point of contact of any of the three tangents (to Γ)

† Needless to say, a bicursal curve is one whose genus (or deficiency) is unity.

that can be drawn through K . If, then, I and J be the two circular points at infinity, we have the equations of residuation

$$[I + J + K] = 0, [3K] = 0 \text{ and } [2\alpha + K] = 0.$$

These automatically lead to

$$[6\alpha] = 0 \text{ and } [4\alpha + I + J] = 0,$$

showing that the point α (on Γ) is sextactic as well as cyclic. Further this α is also a centre of inversion, for the tangent (to Γ) at α meets the line at infinity at precisely the same point (K) as the real asymptote. What holds for α must from symmetry hold also for the points of contact of the other two tangents from K (to Γ). So we arrive at the proposition that *for a bicursal circular cubic Γ with a real inflexion K at infinity, each of the three sextatic points, having K for their common tangential, behave also like a cyclic point and a centre of inversion.*[†]

In the succeeding Article, we shall deal with certain propositions, each of which may in a sense be regarded as the converse of the above proposition.

2. As before, using the letters I, J, K , to denote the two circular points and the real point at infinity on an *unrestricted* circular cubic Γ , we have the permanent equation of residuation

$$[I + J + K] = 0. \quad (1)$$

There are several cases to consider.

Case I. Firstly, suppose that one of the centres of inversion (say, α) behaves like a cyclic point. Then we must have

$$[2\alpha + K] = 0 \quad (2)$$

and

$$[4\alpha + I + J] = 0. \quad (3)$$

Combining (2) and (3) with (1), we get

$$[6\alpha] = 0 \quad (4)$$

and

$$[3K] = 0, \quad (5)$$

showing that the same point α is sextactic and that its tangential K is an inflexion. So Γ must be a cubic of the type considered in Art. 1.

Case II Secondly, suppose that one of the centres of inversion (say, α) is a sextactic point. Then (2) and (4) are the equations to start with and (5) follows as a matter of course, showing that K is an inflexion. Thus we are again led to a cubic of the afore-said type.

Case III. Thirdly, suppose that a point α (on Γ) is both cyclic and sextactic. Then we start with the equations (3) and (4), and combining then with (1), we readily deduce

$$[2\alpha + K] = 0 \text{ and } [3K] = 0.$$

So α must be a centre of inversion and K is an inflexion on the cubic Γ , which must accordingly belong to the above category

[†] It goes without saying that none of the remaining 27-3 or 24 sextactic points of Γ is a cyclic point or a centre of inversion.

Case IV. Fourthly, suppose that three of the centres of inversion (say, $\alpha, \alpha', \alpha''$) of a bicursal circular cubic Γ are collinear. Then we must have

$$[2\alpha + K] = 0, [2\alpha' + K] = 0, [2\alpha'' + K] = 0 \text{ and } [\alpha + \alpha' + \alpha''] = 0.$$

These, taken together, manifestly lead to (5), showing that K is an inflexion and that consequently Γ is a cubic of the above description.

It is hardly necessary to remark that, when three centres of inversion are collinear, the fourth centre of inversion of Γ ,—which is no other than the point of contact of the fourth tangent from K ,—coincides with K itself,—a fact which is otherwise evident from geometrical intuition. The converse result, *viz.* that the existence of a centre of inversion, at infinity implies the collinearity of the other three centres of inversion, ultimately requiring the (circular) cubic Γ to have an inflexion at infinity, easily admits of independent verification.

It is palpably plain that the result of *any* of the four cases I, II, III and IV may be regarded as a converse to that of Art. 1. These isolated results, taken in conjunction with the original proposition of Art. 1, may now be presented together in the following compact form :

A bicursal circular cubic Γ , whose real point at infinity is an inflexion, is definable alternatively :

- (i) *as a (circular) cubic, one of whose centres of inversion is a cyclic point;*
- or (ii) *as a (circular) cubic, one of whose centres of inversion is a sextactic point;*
- or (iii) *as a (circular) cubic, one of whose cyclic points is a sextactic point;*
- or (iv) *as a (circular) cubic, three of whose centres of inversion are collinear;*
- or (v) *as a (circular) cubic, one of whose centres of inversion is at infinity.*

In other words, the five geometrical attributes (i), (ii), (iii), (iv) and (v) must, *if at all*, go hand in hand and any one of them marks out the circular cubic Γ as one, whose real point at infinity is an inflexion.

The circular cubic under discussion in the present article has been uniformly supposed to be *bicursal*. The reader can easily introduce the modifications appropriate to a *unicursal* (circular) cubic.

SECTION II

Analytical treatment based on Cartesian method,

3. If a focal parabola Σ and the associated circle of inversion Π of an *unrestricted* circular cubic Γ be taken in the Cartesian forms

$$y^2 = 4a(x + a) \text{ and } (x - \alpha)^2 + (y - \beta)^2 = \kappa^2, \quad (1)$$

the Cartesian equation of Γ can, without much difficulty, be thrown into the form

$$x(x^2 + y^2) + (2a - \alpha)(x^2 + y^2) + (c^2 - 4a\alpha)x - 4a\beta y + \{2a(\alpha^2 + \beta^2) - c^2\alpha\} = 0, \quad (2)$$

where

$$c^2 \equiv \kappa^2 - \alpha^2 - \beta^2 \quad (3)$$

By elementary algebraic manipulations one can easily turn (2) into the following equivalent form of a similar structure

$$x(x^2 + y^2) + (2a_r - \alpha_r)(x^2 + y^2) + (c_r^2 - 4a_r\alpha_r)x - 4a_r\beta_r y + \{2a_r(\alpha_r^2 + \beta_r^2) - c_r^2\alpha_r\} = 0, \quad (4)$$

provided that the three new sets of constants

$$\left. \begin{aligned} &(\lambda_r, a_r, c_r, \alpha_r, \beta_r) \\ &a_r = a + \lambda_r, \alpha_r = \alpha + 2\lambda_r, \beta_r = a\beta/(a + \lambda_r), \\ &c_r^2 = k_r^2 - \alpha_r^2 - \beta_r^2 = c^2 + 4\lambda_r(2a + \alpha) + 8\lambda_r^2 \end{aligned} \right\} (r = 1, 2, 3) \quad (5)$$

and $\lambda_1, \lambda_2, \lambda_3$ are the three *non-zero* roots † of the biquadratic in λ , viz.,

$$\frac{a^2\beta^2}{a + \lambda} + (a - \lambda)(a + 2\lambda)^2 - 4a\lambda(\alpha + 2\lambda) - c^2\lambda - a(\alpha^2 + \beta^2) = 0. \quad (6)$$

The *algebraic similarity* of the two equations (2) and (4) reveals the fact that the other three focal parabolas ($\Sigma_1, \Sigma_2, \Sigma_3$) and the three associated circles of inversion (Π_1, Π_2, Π_3)* have for their respective pairs of equations

$$\left. \begin{aligned} &(\Sigma_r) \dots y^2 = 4(a + \lambda_r)(x + a_r), \\ &(\Pi_r) \dots (x - \alpha_r)^2 + (y - \beta_r)^2 = k_r^2 \end{aligned} \right\}; (r = 1, 2, 3), \quad (7)$$

and it being understood once for all that the constants ($a_r, \alpha_r, \beta_r, k_r$) are as before given by (5). The common axis ($y = 0$) of the four focal parabolas $\Sigma, \Sigma_1, \Sigma_2, \Sigma_3$ —which are manifestly confocal—will for the sake of brevity be designated as the “axis” of the (circular) cubic Γ and symbolised as Λ .

Plainly the *real* point (K) of Γ at infinity will be an inflexion, provided that the tangent at K , which is no else than the real asymptote (of Γ), viz.,

$$x + 2a - \alpha = 0,$$

meets (2) at a *third* point at infinity. The condition for this to be possible is seen to be

$$\beta = 0, \quad (8)$$

which is precisely the condition for the biquadratic (6) in λ to have one of its non-zero roots (say, λ_3) = $-a$. Then (5) gives $\alpha_3 = \alpha - 2a$ and $\beta_3 = 0/0$ and so *apparently* the corresponding centre of inversion (α_3, β_3) is *indeterminate*. But by *a priori* reasoning this point can be identified with the real point K (at infinity). Further reference to (5) shows that (8) leads to $\beta_1 = 0$ and $\beta_2 = 0$. Thus, as could be expected from other considerations, the *inflectional* character of the point K requires three of the centres of inversion to be on the “axis” Λ and the fourth centre of inversion to move off to infinity.

We may then summarise our conclusions in the form of a proposition:

If one of the four centres of inversion of a bicursal circular cubic Γ happen to lie on its own “axis” Λ , two other centres of inversion must also lie on Λ and the fourth centre of inversion will move off to infinity; and at the same time Γ will have its real point K at infinity for an inflexion. Conversely, every bicursal circular cubic having a real inflexion at infinity, must enjoy the afore-mentioned property.

† For obvious reasons, one of the roots of (6) is 0.

* The traditional relations between the four focal parabolas and their attached circles of inversion can be easily corroborated by reckoning with the equations (1) and (7), provided the relations (5) are properly utilised.

4. Keeping to the notations and conventions of the foregoing article and attending to (8) of the same article, we may represent a circular cubic Γ (with a real inflexion K at infinity) in the Cartesian form :

$$x(x^2 + y^2) + (2a - \alpha)(x^2 + y^2) + (c^2 - 4a\alpha)x + (2a\alpha^2 - c^2\alpha) = 0, \quad (1)$$

where

$$c^2 \equiv k^2 - \alpha^2.$$

Evidently, if I is to be an inflexion on Γ , then J must also be so. Now if I and J are to be inflexions, the two isotropic lines ($y = \pm ix$) drawn through the double focus O ,—which are no else than the tangents to Γ at I and J —must each cut Γ at a third point at infinity. In other words, the necessary and sufficient condition for I (and \therefore for J) to be inflexions on Γ is that the coefficient of x , *vis.*, $c^2 - 4a\alpha$, should be *nil*. Consequently the Cartesian equation of a circular cubic, (bicursal or unicursal), having three inflexions at infinity, may be exhibited in the form

$$F(x, y) \equiv (x + 2a - \alpha)(x^2 + y^2) - 2a\alpha^2 = 0, \quad (\alpha \neq 0). \quad (2)$$

When this cubic Γ is further restricted to have a double point, this (double) point—which must needs be a node*—will have its coordinates satisfying (2) as well as the two equations

$$\frac{\partial F}{\partial x} = 0 \text{ and } \frac{\partial F}{\partial y} = 0. \quad (3), (4).$$

Bearing in mind that the node cannot lie on the asymptote $x + 2a - \alpha = 0$ nor upon the parallel line $x = 0$, we promptly realise that the consistency of the triad of equations (2), (3), (4) demands the fulfilment of the relation

$$2(2a - \alpha)^3 - 27a\alpha^2 \equiv (\alpha + 4a)^2 \cdot (2a - \alpha) = 0,$$

proving that $\alpha = -4a$ or $a/2$. In the former case the node is $(-4a, 0)$; in the latter case, it is $(-a, 0)$. It can be verified on reference to the relations (5) of Art. 3 that the node in either case coincides with a centre of inversion (of Γ), the associated circle of inversion being a *point-circle*. Furthermore, it is a simple affair to verify that the Cartesian equation of Γ , referred to the node as origin, assumes the forms

$$X(X^2 + Y^2) = 2a(8X^2 - Y^2) \text{ and } X(X^2 + Y^2) = \frac{a}{2}(8X^2 - Y^2), \quad (5)$$

according as one or other of the specified points is the node in question. A simple glance at (5) marks out the cubic as a Trisectrix of Maclaurin.

The main results may be summed up as follows :

If a circular cubic Γ , having three inflexions at infinity, possess a double point, then

- (i) this double point, which is clearly a node, must lie on the 'axis' of Γ , (or what is the same thing),
- (ii) it must coincide with a centre of inversion of Γ , the associated circle of inversion reducing to a point,

* For, the existence on a cusp on Γ reduces the number of inflexions to unity.

and (iii) at the same time Γ itself must be a Trisectrix of Maclaurin.†

We may now easily devise a synthetic method of generation of a Trisectrix of Maclaurin viz. :

Let S be the focus and A the vertex of a parabola (Ω). On SA produced take a point B such that $SB = \text{the latus rectum of } \Omega$. Then the envelope of circles, having their centres on the parabola Ω and passing through the fixed point B , is a Trisectrix of Maclaurin.

For obvious reasons, this mode of generation holds for every variety of Trisectrix of Maclaurin.

SECTION III

Alternative Cartesian investigation

5 If the double focus O of a circular cubic Γ (bicursal or unicursal) be chosen as the origin and the line, drawn through O parallel to the asymptote, be chosen as the y -axis ($x = 0$) so that the "axis" of Γ is the x -axis ($y = 0$), the equation of Γ can be readily put in the form

$$(x-\lambda)(x^2+y^2)+ax+by+c=0 \quad (1)$$

Plainly the line ($x = \lambda$) is the real asymptote and the line ($ax+by+c=0$) is the satellite of the line at infinity.

In order that Γ may have its real point K at infinity for an inflexion, the two lines ($x = \lambda$ and $ax+by+c=0$) must be parallel to each other, so that b must vanish. Accordingly the typical equation of a circular cubic, having K for an inflexion, may be taken to be

$$(x-\lambda)(x^2+y^2)+ax+c=0. \quad (2)$$

If, in addition, I or J is to be an inflexion, the line at infinity ($\text{const.} = 0$) must coincide with its own satellite ($ax+c=0$), so that the extra condition ($a=0$) must be fulfilled. Consequently the equation of a circular cubic, having all the three points at infinity, (viz. I, J, K) for inflexions, is

$$(x-\lambda)(x^2+y^2)+c=0, \quad (c \neq 0). \quad (3)$$

Certain characteristic properties of this type of cubic will be discussed in the next article.

6. The notations and conventions of the previous article being kept intact, we may as before start with an *unrestricted* circular cubic in the Cartesian form

$$(x-\lambda)(x^2+y^2)+ax+by+c=0. \quad (4).$$

At the very outset we observe that the polar conic of the double focus O ($0, 0$) w.r.t. Γ , viz.,

$$-\lambda(x^2+y^2)+2(ax+by)+3c=0$$

† Refer to Art. 94, (p. 100) of Ganguly's "Theory of Plane Curves" (Vol. II) (1926) for an alternative demonstration of the known result that a Trisectrix of Maclaurin is definable as a nodal circular cubic with three inflexions at infinity.

will have O for its centre, when and only when

$$a = b = 0, \quad (2)$$

i.e., when and only when Γ has three inflexions at infinity (Art. 5). Other geometrical interpretations can be put upon the pair of conditions (2).

In the first place we observe that the polar line of $O(0, 0)$ w.r.t. Γ viz. the line

$$ax + by + 3c = 0$$

will coincide with the line at infinity (const. = 0), when and only when the same pair of relations (2) is satisfied. So (2) may at pleasure be regarded as the necessary and sufficient condition for the double focus O to be one of the poles of the line at infinity.

Yet another interpretation, involving properties of the Hessian (Γ') of Γ , may be assigned to the conditions (2). For, a point of inflexion of Γ being also a point of inflexion of Γ' , it is geometrically evident that if Γ is to have I, J for inflexions, Γ' must also have I and J for inflexions and so must be a circular cubic. The mode of reasoning being manifestly reversible, we conclude that the two relations (2) represent the necessary and sufficient condition for the Hessian Γ' to be a circular cubic. This result promptly admits of analytic verification.

For, if u_3 denote the group of terms, homogeneous and cubic in x, y , occurring in the Cartesian equation of the Hessian Γ' , we find easily

$$u_3 \equiv \begin{vmatrix} 3x & y & -\lambda x \\ y & x & -\lambda y \\ -\lambda x & -\lambda y & ax + by \end{vmatrix}.$$

Manifestly, u_3 will have

$$x^2 + y^2 \equiv (x + iy)(x - iy)$$

for a factor, if and only if

$$\begin{vmatrix} 3 & \pm i & -\lambda \\ \pm i & 1 & \mp i\lambda \\ -\lambda & \mp i\lambda & a \pm ib \end{vmatrix} = 4(a \pm ib) = 0,$$

i.e., if and only if

$$a = b = 0.$$

This completes the verification of previous statement.

The main results of the present Section may then be presented in the following form :

If a circular cubic Γ (bicursal or unicursal) possesses any one of the four under-mentioned properties. viz.,

- (i) that the two circular points at infinity I, J (and \therefore also the real point K at infinity) be points of inflexion,
- (ii) that the double focus be the centre of its own polar conic,
- (iii) that the double focus be one of the poles of the line at infinity,

- (iv) that the Hessian of Γ be also a circular cubic, then it (Γ) must possess the other three properties, and at the same time its Cartesian equation can, by a suitable choice of Cartesian axes, be thrown into the canonical form:

$$(x-\lambda)(x^2+y^2)+c=0, \quad (c \neq 0).$$

SECTION IV

Special consideration of a unicursal circular cubic with one or more inflexions at infinity

7. The axes of coordinates being chosen as in Art. 5, the Cartesian equation to a circular cubic Γ , having its real point K (at infinity) for an inflexion, may be written as

$$(x-\lambda)(x^2+y^2)+ax+c=0. \quad (1)$$

We shall now work out separately the conditions that Γ may have a double point, in particular, a cusp.

Plainly Γ will have an ordinary double point, provided that the two equations

$$x^3-\lambda x^2+ax+c=0 \quad (2)$$

and

$$3x^2-2\lambda x+a=0 \quad (3)$$

are consistent. The relevant condition for this to be possible is

$$\begin{vmatrix} 0 & 0 & 3 & -2\lambda & a \\ 0 & 3 & -2\lambda & a & 0 \\ 3 & -2\lambda & a & 0 & 0 \\ 0 & 1 & -\lambda & a & c \\ 1 & -\lambda & a & c & 0 \end{vmatrix} = 0.$$

If, in addition, the double point is to be a cusp, the third equation, viz.,

$$3x^2-4\lambda x+\lambda^2=0$$

must hold along with (2) and (3) at the double point. So Γ will have a cusp if and only if

$$a = \frac{\lambda^2}{3} \quad \text{and} \quad c = -\frac{\lambda^3}{27}.$$

Subject to these conditions, the cusp is easily seen to be $(\frac{1}{3}\lambda, 0)$. When, however, this cusp is taken as the origin, the equation to Γ assume the form:

$$X(X^2+Y^2)-\frac{1}{3}\lambda Y^2=0,$$

showing that if a cuspidal circular cubic has a real inflexion at infinity, the cuspidal tangent must be perpendicular to the real asymptote.

8. As noticed in Arts 5 and 6, the Cartesian equation to a circular cubic Γ , having I, J, K for inflexions can, by an appropriate choice of axes, be put in the form

$$(x-\lambda)(x^2+y^2)+c=0, \quad (c \neq 0). \quad (1)$$

Clearly (1) will have a double point, if $c = 4\lambda^3/27$. Further, subject to this condition, the double point is $(\frac{2}{3}\lambda, 0)$. When, however, this double point, (which is after all a node) is taken as the origin, the transformed equation of Γ is

$$X(X^2 + Y^2) = \frac{\lambda}{3} (Y^2 - 3X^2). \quad (2)$$

Thus once again we are led to the *known* result (mentioned in Art. 4), *viz.*, that a unicursal circular cubic with three inflexions at infinity is nothing, if not a Trisectrix of Maclaurin. Joining this result to the conclusions arrived at in Art. 6, we may record the following *equivalent* geometrical definitions of a Trisectrix of Maclaurin, *viz.*,

- (i) *that it is the only nodal circular cubic with three inflexions at infinity;*
 - (ii) *that is the only nodal circular cubic, whose double focus is the centre of its own polar conic, which is none other than a circle;*
 - (iii) *that it is the only nodal circular cubic, whose double focus is one of the poles of the line at infinity,*
- and (iv) *that it is the only nodal circular cubic, whose Hessian is also a circular cubic.*

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ON BATEMAN'S FUNCTION AND AN ALLIED FUNCTION

BY

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1. Bateman (1931) has defined by $K_n(x)$ the function

$$K_n(x) = \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \cos(x \tan \theta - n\theta) d\theta ;$$

when n is an even integer,

$$K_{2n}(x) = \frac{(-1)^n x e^x}{n!} \frac{d^n}{dx^n} (e^{-2x} x^{n-1}) = (-1)^{n-1} M_{n, \frac{1}{2}}(2x). \quad (1)$$

He has deduced the recurrence relations satisfied by $K_n(x)$, viz,

$$(n-2)K_{n-2}(x) - (n+2)K_{n+2}(x) = 4xK'_n(x), \quad (2)$$

$$(n-2)K_{n-2}(x) + (n+2)K_{n+2}(x) + (2n-4x)K_n(x) = 0, \quad (3)$$

$$K'_n(x) + K'_{n+2}(x) = K_n(x) - K_{n+2}(x). \quad (4)$$

The differential equation satisfied by $K_n(x)$ is

$$xK''_n(x) + (n-x)K'_n(x) = 0. \quad (5)$$

The object of the present note is to deduce certain properties of $K_n(x)$ and of another function, allied to $K_n(x)$ which we define by $T_n(x)$, viz,

$$T_n(x) = \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \sin(x \tan \theta - n\theta) d\theta.$$

2. Let us consider the relation (4). Multiplying both sides by $K_n(x)$ first and then by $K_{n+2}(x)$, we get on addition

$$K_n(x)K'_n(x) + K_n(x)K'_{n+2}(x) + K_{n+2}(x)K'_n(x) + K_{n+2}(x)K'_{n+2}(x) = K_n^2(x) - K_{n+2}^2(x),$$

or

$$\frac{1}{2} \frac{d}{dx} (K_n(x) + K_{n+2}(x))^2 = K_n^2(x) - K_{n+2}^2(x).$$

When n is an even integer, we get

$$\frac{1}{2} \frac{d}{dx} \{(K_0 + K_2)^2 + (K_2 + K_4)^2 + \dots + (K_{2n-2} + K_{2n})^2\} = K_0^2 - K_{2n}^2 \quad (6)$$

and when n is an odd integer,

$$\frac{1}{2} \frac{d}{dx} \{(K_1 + K_3)^2 + (K_3 + K_5)^2 + \dots + (K_{2n-1} + K_{2n+1})^2\} = K_1^2 - K_{2n+1}^2. \quad (7)$$

If we now write

$$X = \int_0^\infty e^{-px} K_{2n}(x) dx,$$

then since $K_{2n}(0) = 0$ when $n = 1, 2, \dots$, the equation satisfied by X is

$$\frac{dX}{dp} - \frac{2n-2p}{p^2-1} X = 0,$$

whence

$$X = \frac{c(p-1)^{n-1}}{(p+1)^{n+1}}.$$

Comparing the coefficients of x , we find that the operational representation of $K_{2n}(x)$ is $\phi(p)$, where

$$\phi(p) = \frac{2p(1-p)^{n-1}}{(1+p)^{n+1}}. \quad (8)$$

We have the formula (Van der Pol, 1935)

$$\phi\left(\frac{1}{p}\right) = (-1)^{n-1} \frac{2p(1-p)^{n-1}}{(1+p)^{n+1}} \div \int_0^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}} J_1\{2(xt)^{\frac{1}{2}}\} K_{2n}(t) dt = (-1)^{n-1} K_{2n}(x).$$

Let us write $\frac{1}{2}t^2$ for t and $\frac{1}{2}x^2$ for x . We get

$$\int_0^\infty (xt)^{\frac{1}{2}} J_1(xt) t^{-\frac{1}{2}} K_{2n}(\frac{1}{2}t^2) dt = (-1)^{n-1} x^{-\frac{1}{2}} K_{2n}(\frac{1}{2}x^2), \quad (9)$$

showing that $x^{-\frac{1}{2}} K_{2n}(\frac{1}{2}x^2)$ is self-reciprocal in the Hankel Transform of order 1 if n is an odd integer, and *skew* self-reciprocal if n is an even integer.

3. We have defined $T_n(x)$ as

$$T_n(x) = \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \sin(x \tan \theta - n\theta) d\theta$$

We see that

$$T_n(0) = \frac{2}{\pi n} (\cos \frac{1}{2}n\pi - 1), \quad |T_n(x)| \leq 1.$$

$$T_0(x) = \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \sin(x \tan \theta) d\theta = \frac{2}{\pi} \int_0^\infty \frac{\sin xy}{1+y^2} dy,$$

which is a convergent integral. Since (Watson, 1944)

$$I_{-\nu}(x) - L_{\nu}(x) = \frac{2(\frac{1}{2}x)^{\nu}}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\infty \frac{\sin(xu) du}{(1+u^2)^{\frac{1}{2}+\nu}}$$

we get on putting $\nu = -\frac{1}{2} + \epsilon$,

$$\lim_{\epsilon \rightarrow 0} L\epsilon \Gamma(\epsilon) \{I_{\frac{1}{2}-\epsilon}(x) - L_{-\frac{1}{2}+\epsilon}(x)\} = \frac{2(\frac{1}{2}x)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \int_0^\infty \frac{\sin(xu) du}{(1+u^2)^{\frac{1}{2}}}$$

where $I_0(x)$ is Bessel's Function and $L_0(x)$ is Struve's function with imaginary arguments.

$$T_1(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin xy - y \cos xy}{(1+y^2)^{3/2}} dy = x \{(I_0 - L_0) - (I'_0 - L'_0)\} - \frac{2}{\pi} \quad (10)$$

Let us now evaluate

$$\int_0^{\infty} e^{-x} T_n(x+a) dx,$$

which is equal to

$$\frac{2}{\pi} \int_0^{\infty} e^{-x} dx \int_0^{\frac{1}{2}\pi} \sin \{(x+a) \tan \theta - n\theta\} d\theta,$$

Changing the order of integration which is justifiable by absolute convergence, we have on integration,

$$\int_0^{\infty} e^{-x} T_n(x+a) dx = \frac{1}{2} \{T_{n-2}(a) + T_n(a)\}.$$

This can be written in the form

$$\int_a^{\infty} e^{-x} T_n(x) dx = \frac{1}{2} e^{-a} \{T_{n-2}(a) + T_n(a)\}, \quad (11)$$

Again we notice that

$$\begin{aligned} T'_{n-1}(x) + T'_{n+1}(x) &= \frac{4}{\pi} \int_0^{\frac{1}{2}\pi} \cos(x \tan \theta - n\theta) \sin \theta d\theta \\ &= \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \{\sin(x \tan \theta - \overline{n-1}\theta) - \sin(x \tan \theta - \overline{n+1}\theta)\} d\theta \\ &= T_{n-1}(x) - T_{n+1}(x). \end{aligned} \quad (12)$$

Let us next consider the integral

$$\begin{aligned} &\int_0^{\frac{1}{2}\pi} \frac{d}{d\theta} \{\cos^2 \theta \cos(x \tan \theta - n\theta)\} d\theta \\ &= -2 \int_0^{\frac{1}{2}\pi} \cos \theta \sin \theta \cos(x \tan \theta - n\theta) d\theta - \int_0^{\frac{1}{2}\pi} \cos^2 \theta \sin(x \tan \theta - n\theta) (x \sec^2 \theta - n) d\theta \end{aligned}$$

on simplifying this we find after a bit of calculations the recurrence relation

$$(n-2)T_{n-2}(x) + (n+2)T_{n+2}(x) + (2n-4x)T_n(x) = -8/\pi. \quad (13)$$

Differentiating the above equation and making use of (12) we get the additional recurrence relation

$$(n-2)T'_{n-2}(x) - (n+2)T'_{n+2}(x) = 4xT'_n(x). \quad (14)$$

From (13) and (14) we deduce that

$$(2n-4)T_{n-2}(x) + (2n-4x)T_n(x) = 4xT'_n(x) - 8/\pi. \quad (15)$$

Differentiating and making slight calculations we find that $T_n(x)$ satisfies the differential equation

$$xT''_n(x) + (n-x)T'_n(x) + 2/\pi = 0. \quad (16)$$

4. Let us next multiply (13) by $K_n(x)$ and (3) by $T_n(x)$ and subtract. We get

$$(n-2)(K_n T_{n-2} - K_{n-2} T_n) + (n+2)(K_n T_{n+2} - K_{n+2} T_n) = -8/\pi K_n,$$

or

$$\begin{aligned}
K_{n+2}T_n - K_nT_{n+2} &= \frac{n-2}{n+2}(K_nT_{n-2} - K_{n-2}T_n) + \frac{8}{\pi} \frac{K_n}{n+2} \\
&= \frac{(n-2)(n-4)}{(n+2)n}(K_{n-2}T_{n-4} - K_{n-4}T_{n-2}) + \frac{8}{\pi} \left(\frac{n-2}{(n+2)n} K_{n-4} + \frac{K_n}{n+2} \right) \\
&= \frac{4 \cdot 2}{(n+2)n}(K_4T_2 - K_2T_4) + \frac{8}{\pi n(n+2)} [nK_n + (n-2)K_{n-2} + \dots + 4K_4],
\end{aligned}$$

when n is an even integer. Putting $n = 2$, we get

$$K_4T_2 - K_2T_4 = 2/\pi K_2.$$

Therefore, when n is an even integer,

$$K_{n+2}T_n - K_nT_{n+2} = \frac{8}{\pi(n+2)n} [nK_n + (n-2)K_{n-2} + (n-4)K_{n-4} + \dots + 4K_4 + 2K_2]. \quad (17)$$

Let us next consider the differential equations satisfied by $T_n(x)$ and $K_n(x)$, viz.,

$$T'_n = \left(1 - \frac{n}{x}\right) T_n - \frac{2}{\pi x}, \quad K'_n = \left(1 - \frac{n}{x}\right) K_n.$$

Multiplying the first equation by $K_n(x)$ and the second by $T_n(x)$ and subtracting, we get

$$(K_nT'_n - T_nK'_n) = -\frac{2}{\pi x} K_n, \quad \text{or} \quad \frac{d}{dx} (K_nT'_n - T_nK'_n) = -\frac{2}{\pi x} K_n.$$

Integrating between the limits x and infinity, we get

$$\begin{aligned}
\int_x^\infty \frac{K_n(x)}{x} dx &= \frac{\pi}{2} (K_nT'_n - T_nK'_n) \\
&= \frac{\pi}{8x} \{ (n+2)(K_{n+2}T_n - K_nT_{n+2}) + (n-2)(K_nT_{n-2} - T_nK_{n-2}) \\
&= \frac{1}{nx} \{ nK_n + 2(n-2)K_{n-2} + 2(n-4)K_{n-4} + \dots + 4K_2 \} \quad (18)
\end{aligned}$$

when n is an even integer.

Let us again consider the equations (2) and (18). Let n be an even integer. Multiplying (18) by K_n and (2) by T_n and adding, we get

$$(n-2)(K_nT_{n-2} + K_{n-2}T_n) - (n+2)(K_nT_{n+2} + T_nK_{n+2}) = 4x(K_nT'_n + T_nK'_n)$$

or

$$\begin{aligned}
K_nT_{n+2} + T_nK_{n+2} &= \frac{n-2}{n+2} (K_{n-2}T_n + K_nT_{n-2}) - \frac{4x}{n+2} \frac{d}{dx} (K_nT_n) \\
&= \frac{(n-2)(n-4)}{(n+2)n} (K_{n-4}T_{n-2} + K_{n-2}T_{n-4}) - \frac{4x}{n+2} \frac{d}{dx} (K_nT_n) - \frac{4(n-2)x}{(n+2)n} \frac{d}{dx} (K_{n-2}T_{n-2}) \\
&= \frac{8}{(n+2)n} (K_4T_2 + K_2T_4) - \frac{4x}{(n+2)n} \left[n \frac{d}{dx} (K_nT_n) + (n-2) \frac{d}{dx} (K_{n-2}T_{n-2}) \right. \\
&\quad \left. + \dots + 4 \frac{d}{dx} (K_4T_4) \right].
\end{aligned}$$

Putting $n = 2$, we get

$$K_2 T_4 + K_4 T_2 = -x \frac{d}{dx} (K_2 T_2).$$

Therefore

$$K_n T_{n+2} + T_n K_{n+2} = -\frac{4x}{(n+2)n} \left[n \frac{d}{dx} (K_n T_n) + (n-2) \frac{d}{dx} (K_{n-2} T_{n-2}) \right. \\ \left. + \dots + 4 \frac{d}{dx} (K_4 T_4) + 2 \frac{d}{dx} (K_2 T_2) \right]. \quad (19)$$

5. We shall now find the Fourier series expansion of $\sin(x \tan \theta - n\theta)$ in the interval $(0, 2\pi)$. Let

$$\sin(x \tan \theta - n\theta) = \sum A_m(x) \cos m\theta + \sum B_m(x) \sin m\theta.$$

Then

$$\pi A_m(x) = \int_0^{2\pi} \sin(x \tan \theta - n\theta) \cos m\theta d\theta = \int_0^\pi + \int_\pi^{2\pi}.$$

Let $\theta = 2\pi - \phi$. Then

$$\int_\pi^{2\pi} = - \int_\pi^0 \sin(x \tan \phi - n\phi) \cos m\phi d\phi.$$

Hence $A_m(x) = 0$. Also

$$\pi B_m(x) = \int_0^{2\pi} \sin(x \tan \theta - n\theta) \sin m\theta d\theta = \int_0^\pi + \int_\pi^{2\pi} \\ \int_\pi^{2\pi} = \int_\pi^0 \sin(x \tan \phi - n\phi) \sin m\phi d\phi.$$

Hence

$$\pi B_m(x) = 2 \int_0^\pi \sin(x \tan \phi - n\phi) \sin m\phi d\phi = 2 \int_0^{\frac{1}{2}\pi} + 2 \int_{\frac{1}{2}\pi}^\pi.$$

But

$$\int_{\frac{1}{2}\pi}^\pi = (-1)^{m+n} \int_0^{\frac{1}{2}\pi} \sin(x \tan \theta - n\theta) \sin m\theta d\theta.$$

Therefore

$$\pi B_m(x) = \{1 + (-1)^{m+n}\} \int_0^{\frac{1}{2}\pi} \{\cos(x \tan \theta - \overline{n+m\theta}) - \cos(x \tan \theta - \overline{n-m\theta})\} d\theta \\ = \frac{1}{2}\pi \{1 + (-1)^{m+n}\} [K_{n+m}(x) - K_{n-m}(x)],$$

or

$$B_m(x) = \frac{1}{2}\{1 + (-1)^{m+n}\} [K_{n+m}(x) - K_{n-m}(x)]. \quad (20)$$

6. We shall now evaluate

$$\int_0^\infty e^{-ax} T_n(x) dx$$

which is necessary for operational representation of $T_n(x)$. Let

$$Y_{2n}(a) = \frac{1}{2}\pi \int_0^\infty e^{-ax} T_{2n}(x) dx, \quad (a > 1).$$

Then

$$\frac{2}{\pi} [Y_{2n-2}(a) - Y_{2n}(a)] = \int_0^\infty e^{-ax} [T_{2n-2}(x) - T_{2n}(x)] dx = \int_0^\infty e^{-ax} [T'_{2n-2} + T'_{2n}(x)] dx.$$

Hence

$$\frac{2}{\pi} [Y_{2n-2}(a) - Y_{2n}(a)] = -[T_{2n-2}(0) + T_{2n}(0)] + \frac{2}{\pi} a [Y_{2n-2}(a) + Y_{2n}(a)].$$

But

$$T_{2n}(0) = -\frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \sin 2n\theta d\theta = \frac{2}{\pi} \left\{ \frac{(-1)^n - 1}{2n} \right\}$$

we therefore have

$$\begin{aligned} Y_{2n}(a) &= \frac{1-a}{1+a} Y_{2n-2}(a) + \frac{1}{1+a} \left[\frac{(-1)^{n-1} - 1}{2n-2} + \frac{(-1)^n - 1}{2n} \right] \\ &= \frac{(1-a)^n}{(1+a)^n} Y_0(a) + \left[\frac{1}{1+a} \left(\frac{(-1)^{n-1} - 1}{2n-2} + \frac{(-1)^n - 1}{2n} \right) + \frac{1-a}{(1+a)^2} \right. \\ &\quad \times \left(\frac{(-1)^{n-1} - 1}{2n-4} + \frac{(-1)^{n-1} - 1}{2n-2} \right) + \dots + \frac{(1-a)^{n-1}}{(1+a)^n} \left(\frac{-1-1}{2} \right) \Big], \quad (21) \end{aligned}$$

$$Y_0(a) = \int_0^\infty e^{-ax} dx \int_0^{\frac{1}{2}\pi} \left\{ \sin(x \tan \theta) d\theta \right\}$$

changing the order of integration, we find that

$$Y_0(a) = \frac{\log a}{a^2 - 1}.$$

In a similar manner we can prove that

$$\begin{aligned} Y_{2n+1}(a) &= \frac{(1-a)^n}{(1+a)^n} Y_1(a) - \left\{ \frac{1}{1+a} \left(\frac{1}{2n-1} + \frac{1}{2n+1} \right) \right. \\ &\quad \left. + \frac{1-a}{(1+a)^2} \left(\frac{1}{2n-3} + \frac{1}{2n-1} \right) + \dots + \frac{(1-a)^{n-1}}{(1+a)^n} \left(1 + \frac{1}{3} \right) \right\}, \quad (22) \end{aligned}$$

where

$$Y_1(a) = \frac{1}{2} \pi \int_0^\infty e^{-ax} T_1(x) dx = -\frac{1}{1+a} \left\{ 1 - \frac{1}{(a^2-1)^{\frac{1}{2}}} \tan^{-1}(a^2-1)^{\frac{1}{2}} \right\}$$

since

$$Y_0(a) = \frac{1}{2} \pi \int_0^\infty e^{-ax} T_0(x) dx,$$

it follows that

$$\phi(p) = \frac{p \log p}{p^2 - 1} \div \frac{1}{2} \pi T_0(x)$$

since

$$\phi\left(\frac{1}{p}\right) \div \int_0^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}} J_1\{2(xt)^{\frac{1}{2}}\} \frac{1}{2} \pi T_0(t) dt$$

we easily see that $\frac{1}{2} \pi x^{-\frac{1}{2}} T_0(\frac{1}{2} x^2)$ is self-reciprocal in Hankel Transform of order 1. Similarly $x^{-\frac{1}{2}} [\frac{1}{2} \pi T_1(\frac{1}{2} x^2) + e^{-\frac{1}{2}x}]$ is skew self-reciprocal in Hankel Transform of order 1 and so on.

In conclusion I wish to express my indebtedness to Dr. S. C. Mitra for his help and guidance in the preparation of this paper.

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LUCKNOW

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NOTE ON THE BENDING OF CERTAIN THIN ELASTIC PLATES BY CONCENTRATED LOADS

By

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A Introduction

A simple method of finding the deflection of certain types of elastic plates due to concentrated loads was given by Sen (1934). In this note the same method is applied to solve the problems of thin elastic plates bounded by certain quartic curves in the form of an inverse of an ellipse and an elliptic limaçon with clamped edges and having concentrated loads at the origin of co-ordinates. The chief interest of the solutions lies in the fact that they are obtained in a closed form.

B. Method of Solution

If Z be the constant load per unit area of a thin plate of uniform thickness having the flexural rigidity D , it is known that the normal displacement w satisfies the equation (Love, 1944)

$$D\nabla_1^4 w = Z,$$

where

$$\nabla_1^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.$$

At points where there is no load, this equation becomes

$$\nabla_1^4 w = 0. \quad (1)$$

Suppose now that the boundary of the plate is given by one of the *closed* curves of the family $\eta = \text{constant}$ (or $\xi = \text{constant}$), where ξ and η are a new set of co-ordinates connected with x and y by the relation

$$x + iy = f(\xi + i\eta). \quad (1)$$

If the concentrated load P be situated at points within a certain closed curve s of the family, P is given by the relation

$$\int_s N_s ds = -P, \quad (ii)$$

in which N_s stands for the normal shearing force on an element ds due to the action of the part of the plate lying outside the curve. The value of N_s at any point on the boundary s where $d\nu$ is the element of outward drawn normal, is equal to

$$-D \frac{\partial}{\partial \nu} (\nabla_1^2 w),$$

∇_1^2 standing for

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Hence the relation (ii) can be put as

$$D \int \frac{\partial}{\partial \nu} (\nabla_1^2 w) ds = P, \quad (\text{II})$$

the integration extending over the whole length of the closed curve which belongs to the family $\eta = \text{constant}$ or $\xi = \text{constant}$ as the case may be. If the result of intergration in (II) be independent of the length of the curve, it is apparent that the load can be taken as concentrated over the element of area bounded by the *smallest closed curve* of the family. Moreover, the boundary conditons for a clamped edge are

$$w = \frac{\partial w}{\partial \nu} = 0 \quad (\text{III})$$

at any point on the edge, $d\nu$ being an element of normal at the point. The form of the curve being given by the transformation (i), the problem is to find the solution of the equation (I) which makes the relation (II) independent of the length of the enclosing curve, and satisfies the boundary conditions (III). In the following section this method has been used to determine the normal displacement due to concentrated loads at specified points, the plates being bounded by certain quartic curves. It may be noted in this connection that the problems of bending of uniformly loaded elastic plates bounded by quartic curves of the types discussed here were solved by Sen (1942)

C. Solution of Problems

1. *A plate bounded by an inverse of an ellipse loaded at the centre.* The transformation

$$x + iy = c \sec (\xi + i\eta) \quad (1.1)$$

gives

$$x = \frac{2c \cosh \eta \cos \xi}{\cosh 2\eta + \cos 2\xi}, \quad y = \frac{2c \sinh \eta \sin \xi}{\cosh 2\eta + \cos 2\xi}, \quad (1.2)$$

and

$$\frac{1}{h^2} = \left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \xi} \right)^2 = \frac{2c^2 (\cosh 2\eta - \cos 2\xi)}{(\cosh 2\eta + \cos 2\xi)^2}. \quad (1.3)$$

It is evident that $\eta = \beta$ (a constant) is a closed curve which is the inverse of an ellipse with respect to its centre. Here ξ may have any real value positive or negative, but as the values of x and y are periodic in ξ , it is only necessary to consider values of ξ lying between 0 and 2π . At the origin $\eta \rightarrow \infty$. In the present case the equation (I) reduces to

$$\left[\frac{(\cosh 2\eta + \cos 2\xi)^2}{\cosh 2\eta - \cos 2\xi} \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \right]^2 w = 0. \quad (1.4)$$

On putting

$$w = \frac{P_0 \eta + P_1 \sinh 2\eta + P_2}{\cosh 2\eta + \cos 2\xi}, \quad (1.5)$$

where P_0, P_1, P_2 are constants, it is found that

$$\nabla_1^2 w = \frac{2}{c^2} \left[P_0 \left(\eta - \frac{\sinh 2\eta}{\cosh 2\eta - \cos 2\xi} \right) + P_2 \right]. \quad (1.6)$$

Operating on this expression again by ∇_1^2 it can be easily shown that the equation (1.4) is satisfied.

Taking a particular curve of the family $\eta = \text{constant}$ as the curve s , it is found that

$$\int_s \frac{\partial}{\partial \nu} (\nabla_1^2 w) ds = - \int_0^{2\pi} \frac{\partial}{\partial \eta} (\nabla_1^2 w) d\xi \quad (1.7)$$

since in this case

$$\frac{\partial}{\partial \nu} = -h \frac{\partial}{\partial \eta}, \quad \text{and} \quad ds = \frac{d\xi}{h}.$$

The expression for $\nabla_1^2 w$ obtained in (1.6) being substituted into the result (1.7), the latter becomes

$$-\frac{2P_0}{c^2} \int_0^{2\pi} \left[1 - \frac{\partial}{\partial \eta} \left(\frac{\sinh 2\eta}{\cosh 2\eta - \cos 2\xi} \right) \right] d\xi = -\frac{4P_0\pi}{c^2}. \quad (1.8)$$

As this result is independent of η , the load P assumed in (II) can be taken as concentrated over the small area bounded by the narrowest curve of the family at the centre. Hence from (II)

$$D \left(-\frac{4P_0\pi}{c^2} \right) = P,$$

which gives

$$P_0 = -\frac{Pc^2}{4\pi D}.$$

Then from the boundary conditions (III) which in the present case can be stated as

$$w = \frac{\partial w}{\partial \eta} = 0 \quad \text{when} \quad \eta = \beta,$$

the values of the constants, P_1 and P_2 in (1.5) are obtained in the forms

$$P_1 = \frac{Pc^2}{8\pi D} \operatorname{sech} 2\beta, \quad (1.10)$$

and

$$P_2 = \frac{Pc^2}{8\pi D} [2\beta - \tanh 2\beta]. \quad (1.11)$$

Thus the constants in the expression for w given in (1.5) are completely determined in terms of P . At the centre where $\eta \rightarrow \infty$ it can be easily shown that

$$w \rightarrow \frac{Pc^2}{8\pi D} \operatorname{sech} 2\beta. \quad (1.12)$$

2. *A plate bounded by an elliptic limaçon loaded at the focus.* By using the transformation

$$\xi + i\eta = 2 \sec^{-1} \left[\frac{x + iy}{2c} \right]^{\frac{1}{2}}, \quad (2.1)$$

it is found that $\eta = \beta$ (a constant) represents an inverse of an ellipse with respect to the focus as the pole, that is, an elliptic limaçon. The above transformation gives

$$\frac{4cx}{r^2} = 1 + \cosh \eta \cos \xi, \quad \frac{4cy}{r^2} = \sinh \eta \sin \xi, \quad r = \frac{4c}{\cosh \eta + \cos \xi}, \quad (2.2)$$

and

$$\frac{1}{h^2} = \left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \xi} \right)^2 = \frac{16c^2 (\cosh \eta - \cos \xi)}{(\cosh \eta + \cos \xi)^3}. \quad (2.3)$$

In terms of ξ and η the equation (I) becomes

$$\left[\frac{(\cosh \eta + \cos \xi)^3}{\cosh \eta - \cos \xi} \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \right]^2 w = 0. \quad (2.4)$$

Assuming

$$w = \frac{Q_0 \eta + Q_1 \sinh 2\eta + Q_2}{(\cosh \eta + \cos \xi)^2} \quad (2.5)$$

where Q_0, Q_1, Q_2 are constants. It can be easily verified that the equation (2.4) is satisfied. Moreover, this expression for w gives

$$\nabla_1^2 w = \frac{1}{4c^2} \left[Q_0 \left(\eta - \frac{\sinh \eta}{\cosh \eta - \cos \xi} \right) + \frac{2Q_1 \sinh \eta}{\cosh \eta - \cos \xi} + Q_2 \right]. \quad (2.6)$$

Then for a particular curve s of the family $\eta = \text{constant}$, the integral

$$\begin{aligned} \int_s \frac{\partial}{\partial \nu} (\nabla_1^2 w) ds &= - \int_0^{2\pi} \frac{\partial}{\partial \eta} (\nabla_1^2 w) d\xi \\ &= - \frac{1}{4c^2} \int_0^{2\pi} \left[Q_0 + (2Q_1 - Q_0) \frac{\partial}{\partial \eta} \left(\frac{\sinh \eta}{\cosh \eta - \cos \xi} \right) \right] d\xi = - \frac{Q_0 \pi}{2c^2}. \end{aligned} \quad (2.7)$$

Hence the equation (II) reduces to

$$D \left(- \frac{Q_0 \pi}{2c^2} \right) = P$$

giving

$$Q_0 = - \frac{2c^2 P}{\pi D}. \quad (2.8)$$

Since the result (2.7) is independent of η it can be inferred that the load P is concentrated at the origin which is enclosed by the smallest curve of the family $\eta = \text{constant}$.

The boundary conditions

$$w = \frac{\partial w}{\partial \eta} = 0 \quad \text{when } \eta = \beta$$

will be satisfied if

$$Q_1 = \frac{Pc^2}{\pi D} \operatorname{sech} 2\beta,$$

and

$$Q_2 = \frac{Pc^2}{\pi D} [2\beta - \tanh 2\beta]. \quad (2.9)$$

The values of Q_0 , Q_1 and Q_2 being obtained in terms of the load P , the expression for w given in (2.5) is completely determined. At the origin where $\eta \rightarrow \infty$

$$w \rightarrow \frac{2Pc^2}{\pi D} \operatorname{sech} 2\beta. \quad (2.10)$$

In conclusion, I offer my grateful thanks to Dr. B. Sen for his kind help in the preparation of this paper.

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SOME PROPERTIES OF MACROBERT'S E -FUNCTION

By

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(Communicated by Dr. S. C. Mitra—Received April 4, 1950)

The object of this paper is to deduce some properties of E -Functions of MacRobert and to evaluate some infinite integrals involving the function, with the help of operational calculus.

1. The notation $\varphi(p) \doteq f(x)$ means that $\varphi(p)$ and $f(x)$ are operationally related if

$$\varphi(p) = p \int_0^{\infty} e^{-px} f(x) dx, \quad (1)$$

provided the integral converges.

The E -Function of MacRobert (1942) is defined as

$$E(\alpha, \beta :: x) = \Gamma(\alpha) \int_0^{\infty} e^{-\lambda} \lambda^{\beta-1} \left(1 + \frac{\lambda}{x}\right)^{-\alpha} d\lambda; \quad \mathbf{R}(\beta) > 0 \quad (2)$$

which is equal to $\Gamma(\alpha)\Gamma(\beta)e^{\frac{1}{2}\pi x^{\frac{1}{2}(\alpha+\beta-1)}} W_{\frac{1}{2}-\frac{1}{2}(\alpha+\beta), \frac{1}{2}(\beta-\alpha)}(x)$ when $x \rightarrow \infty$, $E(\alpha, \beta :: x) \rightarrow \Gamma(\alpha)\Gamma(\beta)$ and when $x \rightarrow 0$, $E(\alpha, \beta :: x) \rightarrow 0$, provided $\mathbf{R}(\alpha) > 0$, $\mathbf{R}(\beta) > 0$.

After slight transformations, (2) becomes

$$\frac{1}{p^{\beta-1}} E(\alpha, \beta :: p) \doteq \Gamma(\alpha) x^{\beta-1} (1+x)^{-\alpha} \quad (A)$$

Recurrence Relations: We have

$$\Gamma(\alpha) \left\{ \frac{x^{\beta-1}}{(1+x)^{\alpha}} - \frac{x^{\beta}}{(1+x)^{\alpha+1}} \right\} = \Gamma(\alpha) \frac{x^{\beta-1}}{(1+x)^{\alpha+1}} = \frac{\Gamma(\alpha+1)}{\alpha} \frac{x^{\beta-1}}{(1+x)^{\alpha+1}}$$

or

$$\frac{1}{p^{\beta-1}} E(\alpha, \beta :: p) - \frac{1}{\alpha} \frac{1}{p^{\beta}} E(\alpha+1, \beta+1 :: p) = \frac{1}{\alpha} \frac{1}{p^{\beta-1}} E(\alpha+1, \beta :: p)$$

which on simplification becomes

$$\alpha x E(\alpha, \beta :: x) - x E(\alpha+1, \beta :: x) = E(\alpha+1, \beta+1 :: x). \quad (3)$$

Again we have

$$\begin{aligned} \frac{1}{p^{\beta-2}} E(\alpha, \beta :: p) \doteq \Gamma(\alpha) \frac{d}{dx} \frac{x^{\beta-1}}{(1+x)^{\alpha}} &= \Gamma(\alpha) \left\{ \frac{(\beta-1)x^{\beta-2}}{(1+x)^{\alpha}} - \frac{\alpha x^{\beta-1}}{(1+x)^{\alpha+1}} \right\} \\ &= \frac{(\beta-1)}{p^{\beta-2}} E(\alpha, \beta-1 :: p) - \frac{1}{p^{\beta-1}} E(\alpha+1, \beta :: p) \end{aligned}$$

which on writing $\beta+1$ for β and x for p becomes

$$\alpha x E(\alpha, \beta+1 :: x) = \beta x E(\alpha, \beta :: x) - E(\alpha+1, \beta+1 :: x). \quad (4)$$

Combining (3) and (4), we get

$$(\alpha - \beta)E(\alpha, \beta :: x) = E(\alpha + 1, \beta :: x) - E(\alpha, \beta + 1 :: x) \quad (5)$$

2. We have seen that

$$\frac{1}{p^{\beta-1}} E(\alpha, \beta :: p) = \Gamma(\alpha) p \int_0^\infty e^{-px} \frac{x^{\beta-1}}{(1+x)^\alpha} dx$$

Writing $1/p$ for p , we have

$$pE\left(\alpha, \beta :: \frac{1}{p}\right) = \frac{\Gamma(\alpha)}{p^{\beta-1}} \int_0^\infty \frac{e^{-x/p} x^{\beta-1}}{(1+x)^\alpha} dx$$

we know that

$$\frac{1}{p^{\beta-1}} e^{-x/p} \doteq t^{\dagger(\beta-1)} x^{-\dagger(\beta-1)} J_{\beta-1}\{2(tx)^\dagger\};$$

on interpretation, we have

$$pE\left(\alpha, \beta :: \frac{1}{p}\right) \doteq \Gamma(\alpha) \int_0^\infty \frac{t^{\dagger(\beta-1)} x^{\dagger(\beta-1)}}{(1+x)^\alpha} J_{\beta-1}\{2(tx)^\dagger\} dx.$$

Writing x^\dagger for x and integrating, we find that the right hand side is $2t^{\dagger(\alpha+\beta)-1} K_{\beta-\alpha}(2t^\dagger)$; ($0 < \beta < 2\alpha + \frac{1}{2}$). Hence

$$pE(\alpha, \beta :: 1/p) \doteq 2t^{\dagger(\alpha+\beta)-1} K_{\beta-\alpha}(2t^\dagger) \quad (6)$$

we have the transformation rule, viz., if

$$f(p) \doteq h(x),$$

$$pf\left(\frac{1}{p}\right) \doteq \int_0^\infty J_0\{2(tx)^\dagger\} h(t) dt$$

Hence

$$p \cdot \frac{1}{p} E(\alpha, \beta :: p) \doteq 2 \int_0^\infty t^{\dagger(\alpha+\beta)-1} J_0\{2(tx)^\dagger\} K_{\beta-\alpha}(2t^\dagger) dt$$

which reduces to $\Gamma(\alpha)\Gamma(\beta) {}_2F_1(\alpha, \beta; 1; -x)$. Hence

$$E(\alpha, \beta :: p) \doteq \Gamma(\alpha)\Gamma(\beta) {}_2F_1(\alpha, \beta; 1; -x),$$

the conditions of validity being ($\alpha > 0, \beta > 0$).

3. A form of Parseval's theorem given by Goldstein is that if $\phi_1(p)$ and $\phi_2(p)$ are operationally related to $f_1(x)$ and $f_2(x)$, then

$$\int_0^\infty \frac{\varphi_1(x)f_2(x)}{x} dx = \int_0^\infty \frac{\varphi_2(x)f_1(x)}{x} dx.$$

Let (Van der Pol, 1935)

$$\phi_1(p) = \frac{s^{\beta-1}}{p^{\beta-1}} E\left(\alpha, \beta :: \frac{p}{s}\right); \quad f_1(x) = \Gamma(\alpha)x^{\beta-1}s^{\beta-1}(1+xs)^{-\alpha}$$

$$\varphi_2(p) = p^{-n}e^{-1/p^n}; \quad f_2(x) = x^{\dagger n}J_n(2x^\dagger).$$

Applying the above theorem, we get

$$\begin{aligned}
& \int_0^\infty t^{n-\beta} J_n(2t^{\frac{1}{2}}) E\left(\alpha, \beta :: \frac{t}{z}\right) dt; \quad (n-\beta+\alpha > -1, \beta - \frac{1}{2}n + \frac{1}{2} > 1) \\
& = \int_0^\infty \Gamma(\alpha) e^{-1/t} t^{\beta-n-2} (1+tz)^{-\alpha} dt = \Gamma(\alpha) \int_0^\infty e^{-t} t^{n-\beta+\alpha-1} \left(1 + \frac{t}{z}\right)^{-\alpha} dt \\
& = E(\alpha, n-\beta+\alpha+1; z) z^{-\alpha}.
\end{aligned}$$

Writing $\frac{1}{2}t^2$ for t and $\frac{1}{2}z^2$ for z , we have

$$\int_0^\infty t^{n-2\beta+1} J_n(tz) E(\alpha, \beta :: \frac{1}{2}t^2) dt = z^{-\alpha-1} (\frac{1}{2}z^2)^{2\beta-\alpha-n-1} E(\alpha, n-\beta+\alpha+1; \frac{1}{2}z^2).$$

Let $2\beta = n + \alpha + 1$. Then

$$\int_0^\infty t^{-\alpha} J_n(tz) E(\alpha, \frac{1}{2}(n+\alpha+1) :: \frac{1}{2}t^2) dt = z^{-\alpha-1} E(\alpha, \frac{1}{2}(n+\alpha+1) :: \frac{1}{2}z^2) \quad (n+\alpha > -1; \alpha + \frac{1}{2} > 1) \quad (8)$$

or

$$\int_0^\infty (tz)^{\frac{1}{2}t^{-\alpha-1}} J_n(tz) E(\alpha, \frac{1}{2}(n+\alpha+1) :: \frac{1}{2}t^2) dt = z^{-\alpha-1} E(\alpha, \frac{1}{2}(n+\alpha+1) :: \frac{1}{2}z^2)$$

showing that $z^{-\alpha-1} E(\alpha, \frac{1}{2}(n+\alpha+1) :: \frac{1}{2}z^2)$ is self-reciprocal in the Hankel Transform of order n .

We have proved that

$$p E(\alpha, \beta :: 1/p) \doteq 2t^{\frac{1}{2}(\alpha+\beta)-1} k_{\beta-\alpha}(2t^{\frac{1}{2}}).$$

Also

$$\frac{p}{p+a} \doteq e^{-at}.$$

Therefore

$$\int_0^\infty e^{-at} E\left(\alpha, \beta :: \frac{1}{t}\right) dt = 2 \int_0^\infty \frac{t^{\frac{1}{2}(\alpha+\beta)-1}}{t+a^2} k_{\beta-\alpha}(2t^{\frac{1}{2}}) dt.$$

Let $t = z^2$. The right hand side becomes

$$4 \int_0^\infty \frac{z^{\alpha+\beta-1}}{a^2+z^2} K_{\beta-\alpha}(2z) dz$$

Let $\alpha = \frac{1}{2}$. Then

$$\begin{aligned}
\int_0^\infty e^{-at} E\left(\frac{1}{2}, \beta :: \frac{1}{t}\right) dt &= 4 \int_0^\infty \frac{z^{\beta-\frac{1}{2}}}{a^2+z^2} K_{\beta-\frac{1}{2}}(2z) dz \\
&= \frac{\pi^2 a^{\beta-3/2}}{\sin \pi \beta} \{H_{-(\beta-\frac{1}{2})}(2a) Y_{-(\beta-\frac{1}{2})}(2a) (\beta > 0)\} \quad (9)
\end{aligned}$$

4. We shall now evaluate certain infinite integrals involving E -Function.

Let us consider Bateman's function $K_{2n}(x)$ (Bateman, 1931), whose operational representation is given by

$$K_{2n}(x) \doteq \frac{2p(1-p)^{n-1}}{(1+p)^{n+1}}, \quad (n = 1, 2, \dots).$$

Applying Parseval's Theorem, we get

$$\int_0^\infty x^{-\beta} K_{2n}(x) E(\alpha, \beta :: x) dx = 2\Gamma(\alpha) \int_0^\infty \frac{x^{\beta-1}(1-x)^{n-1}}{(1+x)^{n+\alpha+1}} dx, \quad (\alpha+2 > \beta > 0)$$

we get on expanding the right hand side and integrating term by term

$$\int_0^\infty x^{-\beta} K_{2n}(x) E(\alpha, \beta :: x) dx = \frac{2\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\alpha-\beta+1)}{\Gamma(n+\alpha+1)} \times {}_2F_1[-(n-1), \beta; -(n+\alpha-\beta); -1]. \quad (10)$$

Let us next take the operational representation

$$\frac{p}{(p+1)^m} \doteq \frac{e^{-x} x^m}{\Gamma(m+1)}$$

we get

$$\begin{aligned} \int_0^\infty x^{m-1} e^{-x} E(\alpha, \beta :: x) dx &= \Gamma(\alpha) \int_0^\infty \frac{x^{\beta-1}}{(1+x)^{m+\alpha}} dx \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(m+\alpha-\beta)}{\Gamma(m+\alpha)}; \quad (\beta > 0, m+\alpha-\beta > 0). \end{aligned}$$

Taking the operational representation (Bateman, 1931, p. 828),

$$e^{-x} L_m(2x) \doteq (-1)^m p \frac{(1-p)^m}{(1+p)^{m+1}},$$

we have

$$\begin{aligned} \int_0^\infty x^{-\beta} e^{-x} L_m(2x) E(\alpha, \beta :: x) dx &= (-1)^m \Gamma(\alpha) \int_0^\infty \frac{x^{\beta-1}(1-x)^m}{(1+x)^{m+\alpha+1}} dx \\ &= \frac{(-1)^m \Gamma(\alpha)\Gamma(\beta)\Gamma(m+\alpha-\beta+1)}{\Gamma(m+\alpha+1)} \times {}_2F_1(-m, \beta; -(m+\alpha-\beta); -1), \\ &\quad (\beta > 0, \alpha-\beta > -1). \quad (11) \end{aligned}$$

Let us next take the operational representation (Mitra, 1934)

$$\frac{p(p-1)^n}{(p+1)^{n+3/2}}, \text{ for which the image is } \frac{(-1)^n}{2^{n+1}\Gamma(n+\frac{3}{2})} D_{2n+1}(2x^{\frac{1}{2}})$$

Therefore

$$\begin{aligned} \int_0^\infty x^{-\beta} D_{2n+1}(2x^{\frac{1}{2}}) E(\alpha, \beta :: x) dx &= 2^{n+1} \Gamma(n+\frac{3}{2}) \Gamma(\alpha) \int_0^\infty \frac{x^{\beta-1}(1-x)^n}{(1+x)^{n+\alpha+3/2}} dx \\ &= \frac{2^{n+1} \Gamma(n+\frac{3}{2}) \Gamma(\alpha)\Gamma(\beta)\Gamma(n+\frac{3}{2}+\alpha-\beta)}{\Gamma(n+\frac{3}{2}+\alpha)} \times {}_2F_1(-n, \beta; -(n+\alpha-\beta+\frac{1}{2}); -1), \\ &\quad (\alpha+\frac{3}{2} > \beta > 0). \quad (12) \end{aligned}$$

In conclusion I wish to express my indebtedness to Dr. S. C. Mitra for his help and guidance in the preparation of this paper.

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THE DIOPHANTINE EQUATION $x^2 + y^2 = z^2$

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1. In a footnote of a previous paper (Potts, 1946), I gave the complete solution in rational integers of the diophantine equation

$$x^2 + y^2 = z^2, \quad xyz \neq 0 \quad (1)$$

as

$$\left. \begin{aligned} x &= (m^2 + n^2)[(mr + ns)(p^2 - q^2) + 2(ms - nr)pq], \\ y &= (m^2 + n^2)[2(mr + ns)pq - (ms - nr)(p^2 - q^2)], \\ z &= (m^2 + n^2)(p^2 + q^2), \end{aligned} \right\} \quad (2)$$

where $p^2 + q^2 = r^2 + s^2$.

The complete solution of (1) was given by Rosenthal (1944), and, in fact, he solved the more general equation $x^2 + ay^2 = z^{2n+1}$. The methods used by Rosenthal were not elementary. The purpose of this paper is to give an elementary derivation of the solution (2).

2. Suppose that $(x, y, z) = t$. We set $x = tx_1$, $y = ty_1$, $z = tz_1$. Then (1) becomes

$$x_1^2 + y_1^2 = z_1^2, \quad (x_1, y_1, z_1) = 1. \quad (3)$$

3. The complete solution of

$$x_1^2 + y_1^2 = uv \quad (4)$$

can be obtained by elementary methods (Dickson, 1920, pp. 46-48), and is given by

$$\left. \begin{aligned} x &= k(ac + bd), & u &= k(a^2 + b^2), \\ y &= k(ad - bc), & v &= k(c^2 + d^2). \end{aligned} \right\} \quad (5)$$

By imposing the condition $(x_1, y_1) = 1$ we have $k = 1$, $(a, b) = 1$, and $(c, d) = 1$. Setting $v = z_1^2 = c^2 + d^2$ we have (Dickson, 1920b, p. 169),

$$z_1 = p^2 + q^2, \quad c = p^2 - q^2, \quad d = 2pq, \quad (p, q) = 1.$$

Setting $u = tw_1 = a^2 + b^2$ we get from (4), (5)

$$t = m^2 + n^2, \quad a = mr + ns,$$

$$w_1 = r^2 + s^2, \quad b = ms - nr, \quad (m, n) = 1, \quad (r, s) = 1.$$

Hence the equation

$$x_1^2 + y_1^2 = tw_1 z_1^2, \quad (x_1, y_1, z_1) = 1, \quad (x_1, y_1, w_1) = 1$$

has the complete solution

$$\left. \begin{aligned} x_1 &= (mr + ns)(p^2 - q^2) + 2(ms - nr)pq, \\ y_1 &= 2(mr + ns)pq - (ms - nr)(p^2 - q^2), \\ z_1 &= p^2 + q^2, \quad w_1 = r^2 + s^2, \quad t = m^2 + n^2, \\ (p, q) &= 1, \quad (m, n) = 1, \quad (r, s) = 1. \end{aligned} \right\}$$

To get (3) we set $z_1 = w_1$, i.e.

$$r^2 + s^2 = p^2 + q^2$$

which has the solution

$$r = \frac{1}{2}(a_1a_3 + a_2a_4), \quad p = \frac{1}{2}(a_1a_4 + a_2a_3),$$

$$s = \frac{1}{2}(a_1a_4 - a_2a_3), \quad q = \frac{1}{2}(a_1a_3 - a_2a_4),$$

where, of course, the a 's are to be chosen with proper parities. Combining these results we get the solution (2).

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SPHEROIDAL CONFIGURATION UNDER THE LAW OF DENSITY $\rho = \rho_0(1 - \alpha r^2 - \beta z^2)$

By

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INTRODUCTION

The ellipsoids of revolution or those of unequal axes were shown to be possible forms of equilibrium of a rotating mass of liquid, under self-gravitation, by Maclaurin and Jacobi. With respect to non-homogeneous masses of fluids in equilibrium under rotation certain results are known. Dive (1930) has proved the possibility of the existence of spheroids of heterogeneous masses and variable densities in rotating equilibrium for different types of stratification of matter such as, for instance, when the equi-density surfaces are similar or confocal ellipsoids and not identical with isobars. It is further known that a heterogeneous mass cannot maintain itself in spheroidal or ellipsoidal stratifications either when it rotates with a uniform angular velocity or when the pressure is a function of the density alone [Hamy (1889), Veronnet (1912), Dive (1930), Ghosh (1948, 1949A, 1949B)].

Though it has been shown that a rotating configuration of equilibrium may exist with similar or con-focal distribution of matter, no such actual case has, as yet, been worked out completely. The present paper contains a completely rigorous solution in closed form for a case of a simple law of density-stratifications in similar spheroids. The equi-pressure surfaces are a different family of spheroids and we have assumed both the density and the pressure to vanish on the outer boundary. This model has been studied as a cosmological problem in which not only the density but also the angular velocity vary within the mass. The pressure here is not a function of the density alone.

The model obtained offers certain possibilities of the study of those properties of rotating configurations in equilibrium which are known for homogeneous models under uniform rotation. It is found that, speaking generally, almost all the features of the homogeneous model are retained qualitatively, also in this case, though the deviations in certain features are significant.

For the sake of comparison with the homogeneous model, the Maclaurin's spheroid, we have calculated the changes in $\bar{\omega}^2/2\pi\bar{\rho}$ with the eccentricity, $\bar{\omega}^2$ and $\bar{\rho}$ denoting their average values taken over the volume. We find that, in this case, though the qualitative behavior of $\bar{\omega}^2/2\pi\bar{\rho}$ is the same as in the case of the homogeneous model, the maximum is attained at about the same value of the eccentricity and that the maximum value attained is over 60% higher. We have followed the changes in the dynamical features of the model when it contracts under the restriction of constant

angular momentum and mass. It is found that the angular velocity increases monotonically and reaches an upper limit as the eccentricity, e , tends to 1. The central density also increases monotonically. The equatorial radius has been supposed to start from a very large value and is found to tend to a limiting minimum value as $e \rightarrow 1$. Besides as the model contracts it must also flatten.

In the second part of the paper the constituent matter of the model has been supposed to be collected from a state of infinite diffusion with zero energy but with a definite total mass and a definite angular momentum. For the latter we have to assume the vanishing of the angular velocity in a suitable way in the state of infinite diffusion. The whole change is supposed to take place slowly under adiabatic conditions, that is, without exchange of energy with any other system. At any contracted stage the kinetic, the potential and the heat energies of the configuration have been calculated on the assumption that the material obeys the perfect gas law. The total energy, obtained by summing up the three types of energy is found to be negative. The corresponding loss has been attributed to the loss of energy suffered by the model due to radiation. The calculation of the total energy also shows that the model remains thermodynamically stable even when γ , the ratio of the specific heats, is less than $\frac{4}{3}$. Rotation thus appear also to influence thermodynamic stability, in general, increasing it.

Part I.

Sec. 1 A.

THE CONFIGURATION OF EQUILIBRIUM

The equations of steady motion of a fluid, in cylindrical co-ordinates, with the z -axis as the axis of symmetry, are

$$\omega^2 r = - \frac{\partial \Phi}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} \quad (1.1)$$

$$0 = - \frac{\partial \Phi}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial z} \quad (1.2)$$

the angular velocity ω and the density ρ being, in general, variable. The function Φ representing the gravitational potential must satisfy at every point inside the fluid the equation

$$\nabla^2 \Phi = -4\pi\rho. \quad (1.3)$$

The above three equations give

$$\nabla^2 \left(\frac{1}{\rho} \frac{\partial p}{\partial z} \right) + 4\pi \frac{\partial p}{\partial z} = 0 \quad (1.4)$$

This condition of integrability supplies at the same time the necessary type of connection between the pressure and density in order that the equilibrium configuration may be possible. We shall call it the condition of consistency of p and ρ .

In a previous paper (Ghosh, 1949A) we have shown that 'when the density-stratifications are in similar ellipsoids and the pressure is a function of the density alone' the distribution of density should be given by

$$\rho = \rho_0(1 - \alpha r^2 - \beta z^2) \quad (1.5)$$

where

$$\alpha = 1/a^2; \beta = 1/c^2 \quad (1.5a)$$

and the pressure-density relation, satisfying eqn. (1.4), must be of the form

$$p = \kappa \rho^3 + \kappa' \rho^2.$$

But, as was shown, in this case equilibrium under self-gravitation is impossible.

We propose now to show that, for the density-distribution given by (1.5), equilibrium under self-gravitation is possible if the distribution is baroclinic, the pressure being given by

$$p = \kappa \rho^3 + \kappa' \rho^2(1 - m r^2). \quad (1.6)$$

When the density and the pressure are given by (1.5) and (1.6) respectively, the consistency condition (1.4) requires

$$3k\rho_0(2\alpha + 3\beta) + 4k'm - 2\pi = 0. \quad (1.7)$$

It must be noted that the density and the pressure both vanish on the boundary of the fluid-body given by

$$1 - \alpha r^2 - \beta z^2 = 0; \quad (1.8)$$

the density is positive inside the mass of the fluid and diminishes outwards. The pressure, also, is positive everywhere inside, if,

$$m \leq \alpha. \quad (1.9).$$

From (1.1) and (1.2) we obtain,

$$\frac{\partial}{\partial z}(\omega^2 r) = -\frac{1}{\rho^2} \left[\frac{\partial p}{\partial r} \frac{\partial \rho}{\partial z} - \frac{\partial p}{\partial z} \frac{\partial \rho}{\partial r} \right]. \quad (1.10)$$

Hence, from (1.5) and (1.6) we have

$$\frac{\partial \omega^2}{\partial z} = 2mk' \frac{\partial \rho}{\partial z},$$

which, on partial integration, leads to

$$\omega^2 = 2mk'\rho + F(r), \quad (1.11)$$

$F(r)$ being an arbitrary function of r . Equation (1.11) gives the distribution of the angular velocity within the fluid mass.

Equations (1.1) and (1.2) give

$$\Phi = \int \left[\frac{dp}{\rho} - \omega^2 r \cdot dr \right]$$

which, by (1.6) and (1.11), leads to

$$\Phi = \frac{3}{2}k\rho^2 + 2k'\rho(1 - mr^2) - \frac{1}{2} \int F(r^2) \cdot d\tau^2. \quad (1.12)$$

This potential field is necessary for the equilibrium of the fluid.

Sec. IB.

THE POTENTIAL DUE TO SELF-GRAVITATION

For the distribution (1.5) with the boundary (1.8), the Newtonian potential due to self-gravitation is given by (Hobson, 1896)

$$V = \frac{\pi\rho_0 a^2 c}{2} \int_0^\infty \frac{d\theta}{(a^2 + \theta)(c^2 + \theta)^{\frac{1}{2}}} \left(1 - \frac{r^2}{a^2 + \theta} - \frac{z^2}{c^2 + \theta}\right)^{\frac{1}{2}} \quad (1.13)$$

Putting (1.13) in the form

$$\frac{2V}{\pi\rho_0 a^2 c} = [A_0 - 2A_1 r^2 - 2A_2 z^2 + 2A_3 r^2 z^2 + A_4 r^4 + A_5 z^4] \quad (1.14)$$

we have the values of $A_0, A_1, A_2, A_3, A_4, A_5$ given by eqns. (3.17) of the paper referred to before (Ghosh, 1949A).

We shall now investigate the condition under which the potential Φ is due entirely to the self-gravitation of the spheroidal configuration. From (1.14), we have

$$\left. \begin{aligned} \frac{\partial V}{\partial r} &= -2\pi\rho_0 a^2 c r [A_1 - A_4 r^2 - A_5 z^2] \\ \frac{\partial V}{\partial z} &= -2\pi\rho_0 a^2 c z [A_2 - A_3 r^2 - A_5 z^2] \end{aligned} \right\} \quad (1.15)$$

and from (1.12) we have

$$\frac{\partial \Phi}{\partial r} = 3k\rho \frac{\partial \rho}{\partial r} + 2k' \frac{\partial \rho}{\partial r} - 2mk' \left(r^2 \frac{\partial \rho}{\partial r} + 2r\rho \right) - F(r^2)r.$$

If, for the moment, we put

$$F(r^2) = 2\rho_0(B_1 + B_2 r^2) \quad (1.16)$$

we get

$$\frac{\partial \Phi}{\partial r} = -2\rho_0 r [(3k\rho_0 + 2k')\alpha + 2mk' + B_1] - \alpha r^3 \{3k\rho_0 \alpha + 4mk' - B_2\} - \beta z^2 \{3k\rho_0 \alpha + 2mk'\}; \quad (1.17)$$

also,

$$\frac{\partial \Phi}{\partial z} = -2\rho_0 \beta z [(3k\rho_0 + 2k') - (3k\rho_0 \alpha + 2mk')r^2 - 3k\rho_0 \beta z^2]. \quad (1.18)$$

The condition for self-gravitation is obtained by identifying V with Φ . A comparison of (1.17) with the first equation of (1.15) justifies our assumption (1.16). The identification of the corresponding derivatives of V and Φ gives

$$3k\rho_0 \alpha + 2k'(\alpha + m) + B_1 = \pi a^2 c A_1 \quad (1.19a)$$

$$3k\rho_0 \alpha + 4mk' - B_2 = \pi a^4 c A_2 \quad (1.19b)$$

$$3k\rho_0 + 2mk' = \pi a^2 c^3 A_3 \quad (1.19c)$$

$$3k\rho_0 \alpha + 2k' = \pi a^2 c^3 A_4 \quad (1.19d)$$

$$3k\rho_0 \alpha + 2mk' = \pi a^2 c^3 A_5 \quad (1.19e)$$

$$3k\rho_0 = \pi a^2 c^5 A_5 \quad (1.19f)$$

We may now proceed to an analysis of these equations. Of the six equations (1.19) we find that the third and the fifth are identical. So that there are only five independent equations. We have, in addition, equation (1.7). But, as may be expected, equation (1.7) is derivable from the above set and as such is not an equation independent of (1.19). Thus we have altogether *five* independent equations for a given set of values of a and c . There are, however, six parameters, ρ_0 , k , k' , m , B_1 and B_2 . But equations (1.19) show that k and ρ_0 appear in them always as the combination $k\rho_0$. Hence, from (1.19) we can exactly determine the quantities $k\rho_0$, k' , m , B_1 and B_2 but not k and ρ_0 separately. Solving (1.19) we find

$$3k\rho_0 = \pi a^2 c^5 \int_0^\infty \frac{d\theta}{(a^2 + \theta)(c^2 + \theta)^{5/2}} \quad (1.20a)$$

$$2mk' = \pi c^3 (a^2 - c^2) \int_0^\infty \frac{\theta \cdot d\theta}{(a^2 + \theta)^2 (c^2 + \theta)^{5/2}} \quad (1.20b)$$

$$2k' = \pi a^2 c^3 \int_0^\infty \frac{\theta \cdot d\theta}{(a^2 + \theta)(c^2 + \theta)^{5/2}} \quad (1.20c)$$

$$B_1 = \pi c (a^2 - c^2) \int_0^\infty \frac{\theta^2 d\theta}{(a^2 + \theta)^2 (c^2 + \theta)^{5/2}} \quad (1.20d)$$

$$B_2 = -\pi c (a^2 - c^2)^2 \int_0^\infty \frac{\theta^2 d\theta}{(a^2 + \theta)^3 (c^2 + \theta)^{5/2}} \quad (1.20e)$$

The above integrals can all be evaluated; their values have been found and used in a subsequent section [Eqns. (2.7)]. Equations (1.20) show that B_1 is positive, B_2 is negative but $B_1 + B_2$ is positive. Also from (1.11) and (1.16) we have

$$\omega^2 = 2mk'\rho + 2\rho_0(B_1 + B_2 r^2); \quad (1.21)$$

hence, ω^2 is positive everywhere inside and on the surface (where ρ vanishes) ω^2 attains the largest value $2\rho_0 B_1$ at the pole and the smallest value $2\rho_0(B_1 + B_2)$ at the equatorial belt. This is in agreement with Dive's result (Dive, 1930, p. 64) where he proves that the density-distribution being in similar spheroids, the angular velocity on any layer of equal density increases from the equator towards the pole. It may be noted that the distribution of ω^2 on the surface is parabolic and there is a maximum of ω^2 .

The above equations are also in agreement with (1.9) for, as

$$2k'(m - \alpha) = \pi c^3 \int_0^\infty \frac{\theta \cdot d\theta}{(a^2 + \theta)(c^2 + \theta)^{5/2}} \left\{ \frac{a^2 - c^2}{a^2 + \theta} - 1 \right\} \leq 0, \quad m \leq \alpha$$

as $2k'$ is essentially positive. It can be shown from the above integral that as $c \rightarrow 0$, $m \rightarrow 0$ or $a \rightarrow \infty$ and as $c \rightarrow 1$, $m \rightarrow \alpha$.

Thus equations (1.5), (1.6), (1.21) represent the solution for equilibrium under self-gravitation of a mass of heterogeneous fluid rotating steadily in the form of an oblate spheroid about its axis of symmetry, the density being distributed in spheroids (1.5) similar to the outer boundary and the angular velocity being variable and distributed

(21). The density, the pressure and the angular velocity, ω , the two former vanishing on the surface.

Sec. 2. THE FUNCTION $\bar{\omega}^2/2\pi\bar{\rho}$.

The study of the uniformly rotating homogeneous model such as Maclaurin's spheroid and Jacobis ellipsoid has shown that the ratio $\omega^2/2\pi\rho$ plays an important part in its theory, being connected with the stability of the configuration. It is known that for the Maclaurin's spheroid this parameter $\omega^2/2\pi\rho$ has a single maximum 225 . . . occurring for the eccentricity $e = .93$. . . , of the meridian elliptic section. We propose, now, to undertake a similar study with respect to our present model. As the angular velocity and the density are both variable in this case, we shall consider, for the sake of comparison, only the volume averages of both ω^2 and ρ . Thus we define

$$\bar{\omega}^2 = \int \omega^2 dv / \int dv \quad (2.1)$$

and

$$\bar{\rho} = \int \rho dv / \int dv \quad (2.2)$$

where the integrations are extended over the whole volume of the fluid. From (2.2), we have

$$\bar{\rho} = \frac{2}{3} \rho_0 \quad (2.3)$$

and

$$\bar{\omega}^2 = 2mk' \cdot \frac{2}{3} \rho_0 + 2\rho_0 B_1 + \frac{2}{3} \rho_0 \cdot 2B_2 \quad (2.4)$$

whence

$$\bar{\omega}^2/\bar{\rho} = 2mk' + 5B_1 + 2B_2. \quad (2.5)$$

Now, a consideration of the equations (1.20) shows that $3k\rho_0/a^2$, $2k'/a^2$, $2mk'$, B_1 and B_2 depend only on the ratio of a and c and not on their absolute values.

Hence (2.5) shows that $\bar{\omega}^2/\bar{\rho}$ also depends only on the eccentricity, e , of the boundary, exactly as in the case of the Maclaurin's spheroid. It is possible to obtain the five quantities mentioned above in terms of the eccentricity alone. Instead of the eccentricity it is more convenient and at the same time neater to obtain the same in terms of x , where we put

$$e^2 = 1/(1+x^2), \quad (2.6)$$

so that,

$$\text{as } e \rightarrow 0, x \rightarrow \infty \text{ and as } e \rightarrow 1, x \rightarrow 0. \quad (2.6a).$$

We have the following results

$$3k\rho_0/a^2 = \frac{2}{3} \pi x^2 [(1-3x^2) + 3x^2(x \cot^{-1}x)] \quad (2.7a)$$

$$2k'/a^2 = \frac{2}{3} \pi x^2 [(2+3x^2) - 3(1+x^2)(x \cot^{-1}x)] \quad (2.7b)$$

$$2mk' = \pi x^2 [(5x^2 + \frac{4}{3}) - (5x^2 + 3)(x \cot^{-1}x)] \quad (2.7c)$$

$$B_1 = \pi x [(1+5x^2)(1+x^2)(\cot^{-1}x) - x(5x^2 + \frac{1}{3})] \quad (2.7d)$$

$$B_2 = \frac{1}{4} \pi x [(\frac{5}{3} + 35x^2)x - (3+30x^2+35x^4)(\cot^{-1}x)] \quad (2.7e)$$

It will be useful to note that (i) when $e \rightarrow 1$, i.e., $x \rightarrow 0$.

$$\left. \begin{aligned} \frac{3k\rho_0}{a^3} &\rightarrow \frac{2\pi}{3}x^2; \quad \frac{2k'}{a^2} \rightarrow \frac{4\pi}{3}x^2; \quad 2mk' \rightarrow \frac{4\pi}{3}x^2 \\ B_1 &\rightarrow \frac{\pi^2}{2}x; \quad B_2 \rightarrow -\frac{3\pi^2}{8}x \end{aligned} \right\} \quad (2.8)$$

(ii) when $e \rightarrow 0$, i.e., $x \rightarrow \infty$

$$\left. \begin{aligned} \frac{3k\rho_0}{a^3} &\rightarrow \frac{2\pi}{5}; \quad \frac{2k'}{a^2} \rightarrow \frac{4\pi}{15}; \quad 2mk' \rightarrow \frac{4\pi}{35} \cdot \frac{1}{x^2} \\ B_1 &\rightarrow \frac{16\pi}{105} \cdot \frac{1}{x^2}, \quad B_2 \rightarrow -\frac{16\pi}{315} \cdot \frac{1}{x^4} \end{aligned} \right\} \quad (2.9).$$

With values substituted from (2.7), eqn. (2.5) takes the form

$$\bar{\omega}^2/2\pi\bar{\rho} = \frac{1}{12}[8(5x^4 + 24x^2 + 7)(\text{reot}^{-1}x) - (15x^4 + 67x^2)]. \quad (2.10).$$

From this it is clear that when $x \rightarrow 0$, $\bar{\omega}^2/2\pi\bar{\rho} \rightarrow 0$; when $x \rightarrow \infty$, $\bar{\omega}^2/2\pi\bar{\rho} \rightarrow 0$.

TABLE 1

(Present model)

| x | e | $\bar{\omega}^2/2\pi\bar{\rho}$ |
|----------|-------|---------------------------------|
| ∞ | 0 | 0 |
| 10 | 0.09 | 0.0043 |
| 5 | 0.196 | 0.0162 |
| 2 | 0.447 | 0.1227 |
| 1 | 0.707 | 0.2352 |
| 0.8 | 0.78 | 0.2889 |
| 0.7 | 0.81 | 0.3178 |
| 0.6 | 0.85 | 0.3452 |
| 0.5 | 0.89 | 0.3682 |
| 0.4 | 0.92 | 0.3800 |
| 0.3 | 0.957 | 0.3700 |
| 0.2 | 0.98 | 0.3296 |
| 0.1 | 0.99 | 0.2017 |
| 0.01 | 0.998 | 0.0275 |
| 0 | 1 | 0 |

TABLE 2

(Maclaurin's Spheroid)

| e | $\omega^2/2\pi\rho$ |
|------|---------------------|
| 0 | 0 |
| 0.1 | 0.0027 |
| 0.2 | 0.0107 |
| 0.4 | 0.0486 |
| 0.7 | 0.1887 |
| 0.8 | 0.1816 |
| 0.9 | 0.2203 |
| 0.92 | 0.2241 |
| 0.93 | 0.2247 |
| 0.94 | 0.2239 |
| 0.95 | 0.2213 |
| 0.96 | 0.2160 |
| 0.97 | 0.2063 |
| 0.98 | 0.1890 |
| 0.99 | 0.1551 |
| 1 | 0 |

Thus $\bar{\omega}^2/2\pi\bar{\rho}$ vanishes at both the limits $e \rightarrow 1$ and $e \rightarrow 0$. The second result can be demonstrated by expanding the right-hand-side of (2.10) in powers of $\frac{1}{x}$ which gives

$$\frac{\bar{\omega}^2}{2\pi\bar{\rho}} = \frac{1}{12} \left[\frac{184}{85} \cdot \frac{1}{x^2} - \frac{464}{3.5.7} \cdot \frac{1}{x^4} + \frac{40}{11} \cdot \frac{1}{x^6} - \dots \right] \quad (2.11)$$

for $x > 1$. The general behaviour of $\bar{\omega}^2/2\pi\bar{\rho}$ can, however, be studied from the numerical values shown in Table 1.

The values in Table 2 show the variation of $\bar{\omega}^2/2\pi\bar{\rho}$ with eccentricity for the Maclaurin's spheroid and are given here for comparison.

From a study of the two tables it becomes clear that in the case of both the models $\bar{\omega}^2/2\pi\bar{\rho}$ vanishes at both ends *viz.*, $e \rightarrow 0$ and $e \rightarrow 1$, and possesses a maximum for a value of the eccentricity which is pretty large. The eccentricities corresponding to the maximum value are very nearly the same. But, whereas the maximum value attained for the homogeneous model is 0.225 that for the present model is 0.37, which is over 60% higher.

Sec. 3.

CONFIGURATIONS OF EQUILIBRIUM FOR VARIATIONS IN ρ_0 , $\bar{\omega}$ AND a

The model under consideration has, in a way, three degrees of freedom. For, given any set of values of a and c (considered as parameters) the values of $(k\rho_0)$, k' , m , B , and B_2 can be determined but k or ρ_0 are not individually ascertained. Hence we are at liberty to choose either k or ρ_0 , so that the other may adjust itself according to (1.20a). Let us suppose that the rotating fluid-mass under consideration, for some reason or other, undergoes a change of form represented by a continuous variation in eccentricity. The mass and the angular momentum of the fluid must be conserved and hence the number of degrees of freedom will be reduced to one only. We take the eccentricity to be this changing parameter and represent it through x .

Now, the mass, M is given by

$$M = \frac{8}{15} \pi \rho_0 a^2 c. \quad (3.1)$$

The angular momentum, A , is given by

$$A = \int \omega r^2 \rho dv.$$

To avoid very difficult integrations we approximate this by

$$\bar{\omega} \int r^2 \rho dv. \quad (3.2)$$

where ω , again, replaces $(\bar{\omega}^2)^{\frac{1}{2}}$. We have thus the approximate relation

$$A = \bar{\omega} \cdot \frac{8}{7} \cdot M a^2. \quad (3.3)$$

These approximations will not vitiate the general arguments which follow. Further, in (2.10) we put

$$\bar{\omega}^3/2\pi\bar{\rho} = E(x). \quad (3.4)$$

so that,
$$E(x) = \frac{1}{12}[3(5x^4 + 24x^2 + 7)(x \cot^{-1} x) - (15x^4 + 87x^2)]. \quad (3.4)$$

From (3.4) and (3.2) together with (2.3) we have

$$\bar{\omega}^3 \approx (\bar{\omega})^2 = \frac{4}{3} \pi \rho_0 E(x). \quad (3.6)$$

Also, (3.1) can be written as

$$M = \frac{8}{15} \cdot \pi \rho_0 a^3 \frac{x}{(1+x^2)^{\frac{1}{2}}}. \quad (3.7)$$

(i) *Study of ρ_0 .* Eliminating $\bar{\omega}$ and a from (3.3), (3.6) and (3.7) we have

$$\rho_0 = L \{E(x)\}^3 \left(1 + \frac{1}{x^2}\right)^3 \quad (3.8)$$

where

$$L = \left(\frac{3}{49}\right)^3 \cdot \frac{15}{\pi} \cdot \frac{M^{10}}{A^4}. \quad (3.8a)$$

For all variations in x , under the circumstances considered, M and A remain constant and hence L behaves as a constant. From (3.8) it is easy to see that

$$\lim_{x \rightarrow 0} \frac{E(x)}{x} = \frac{7\pi}{8} \quad (3.9)$$

and from (2.11), that

$$\lim_{x \rightarrow \infty} E(x) = 0. \quad (3.10)$$

Hence, from (3.8) we find that

$$\lim_{x \rightarrow \infty} (\rho_0/L) = 0. \quad (3.11)$$

and

$$\lim_{x \rightarrow 0} (\rho_0/L) \rightarrow \infty. \quad (3.12)$$

In other words, at the spherical end of the series of configurations represented by varying e , ρ_0/L , i.e., also $\bar{\rho}/L$ attains an infinitesimally small value, but it increases without limit with increasing eccentricity i.e., for increasing flattening. It is to be noted that by reason of (2.3) ρ_0 , the central density, can be taken to represent the mean density of the configuration.

It may be noted further that we can not show from (3.4), (3.3) and (3.8a) that

$$\rho_0^4 = \left(\frac{3.5}{4.49 \cdot \pi}\right)^3 \cdot \frac{15}{\pi} \cdot \frac{M^{10}}{I^4} \cdot \left(1 + \frac{1}{x^2}\right)^3 \quad (3.13)$$

where I represents the moment of inertia of the body and so,

$$A = I_{\bar{\omega}}. \quad (3.13a)$$

Hence

$$\lim_{x \rightarrow \infty} = \rho_0 \frac{15}{\pi(3.49)^{\frac{1}{2}}} \cdot \frac{M^{5/2}}{I_0^{3/2}} \quad (3.14)$$

when I_0 , the limiting value of the moment of inertia in the spherical case does not vanish.

The general nature of the variation in $\bar{\rho}/L$ with e , i.e., of the increase of the mean density of the configuration with eccentricity is shown in Table 3.

TABLE 3

| x | $\rho_0/L = \frac{3}{2}\bar{\rho}/L$ | ω^2/L' |
|----------|--------------------------------------|---------------|
| ∞ | 0 | 0 |
| 1 | 0.04 | 0.0094 |
| 0.5 | 1.25 | 0.4602 |
| 0.4 | 2.89 | 1.0982 |
| 0.3 | 15.196 | 7.1393 |
| 0.2 | 24.269 | 8.0058 |
| 0.1 | 81.608 | 16.32 |
| 0.01 | ~ 800 | 22.00 |
| 0 | ∞ | 56.14 |

(ii) *Study of ω* Eliminating ρ_0 from (3.6) and (3.8) we have,

$$\frac{\omega^2}{L'} = \{E(x)\}^4 \left(1 + \frac{1}{x^2}\right)^2 \quad (3.15)$$

where

$$L' = \frac{4\pi}{3} \cdot L. \quad (3.16)$$

As $x \rightarrow \infty$, ($e \rightarrow 0$), $\bar{\omega}^2/L' \rightarrow 0$. Also, as $x \rightarrow 0$ ($e \rightarrow 1$), $\bar{\omega}^2/L' \rightarrow$ a finite limit, given by

$$\lim_{x \rightarrow 0} \frac{\bar{\omega}^2}{L'} = \lim_{x \rightarrow 0} \left\{ \frac{E(x)}{x} \right\}^4 = \left(\frac{7\pi}{8} \right)^4. \quad (3.17)$$

Thus, whereas at the spherical end of the series $\bar{\omega}^2/L'$ vanishes, it attains an upper limit when the model has very nearly turned into a flat disc. The nature of the variations of $\bar{\omega}^2/L'$ midway between these two extremes is shown in Table 3.

A joint study of the values of $\bar{\rho}/L$ and $\bar{\omega}^2/L'$ clearly shows the mechanism which controls the behaviour of $\bar{\omega}^2/2\pi\bar{\rho}$ described in Section 2 in the present case. Starting from the spherical shape, as the flattening proceeds ω^2 at first increases at a rate higher than that of $\bar{\rho}$; then, later, $\bar{\rho}$ overtakes and ultimately far surpasses the rate of increase of $\bar{\omega}^2$.

(iii) *Study of a* . Since ω attains a maximum as $e \rightarrow 1$, it is clear from (3.3) that a , then, tends to the minimum value given by

$$a_{min} = \left(\frac{7A}{2M} \right)^{\frac{1}{2}} \cdot \frac{1}{(L')^{\frac{1}{2}}} \cdot \left(\frac{8}{7\pi} \right)^2 \quad (3.18)$$

which follows from (3.17). From Table 3 we can also study the changes in a for variations of e in the present case.

The limiting case $e \rightarrow 0$ ($x \rightarrow \infty$) can be given two different interpretations. The above discussion shows that in that case $\bar{\omega} \rightarrow 0$. Eqn. (3.3) may be satisfied either by taking $A = 0$ (non-rotating configuration) when a may have a finite limit corresponding to the limit of $(A/\bar{\omega})_{e=0}$ or by assuming A to have a finite value while $a \rightarrow \infty$, corresponding to an infinite configuration (with finite angular momentum), and it may be easily seen that the density $\bar{\rho}$ in this latter limiting case vanishes. The first case corresponds to a non-rotating spherical configuration in equilibrium [eqns. (1.20) and (1.21)] and the second to the limiting configuration of infinite diffusion of a given mass of matter with the contemplated density law and a definite amount of angular momentum, as its dimension increases without limit but $\bar{\omega} \rightarrow 0$.

Part II

Sec. 4. ENERGY

So far we have considered our material to be only a fluid with baroclinic distribution—the pressure not being entirely determined by the density. We have seen that the level surfaces of density are a family of ellipsoids while the equi-pressure surfaces are ellipsoids of a different family. We now assume that our fluid obeys the equation of state of perfect gases given by (4.6). The pressure and the density distributions inside the mass being known, this law determines the distribution of temperature within the mass uniquely.

In the manner of Section 2 we shall trace the changes in the energy of the configuration in a given mass with a given angular momentum both of which remain constant in course of the changes. We shall assume that the gas particles start from a state of infinite diffusion and are slowly gathered into the form which we have studied in our previous sections. This process of collection may be assumed to take place infinitely slowly under adiabatic condition, that is, without any exchange of energy with a different system. Originally, in the state of infinite diffusion we may consider the energy of the system to be zero though a finite angular momentum may be attributed to the mass. This is possible as the kinetic energy is proportional to the square of the angular velocity while the angular momentum is proportional to its first power and the angular velocity in the case vanishes. As the gas is collected into a finite volume adiabatically it will acquire gravitational energy, kinetic energy of rotation and heat energy. The sum of these three types of energies cannot exceed the original zero value. A negative sum will mean that the mass has lost this amount by radiation. We shall next investigate how the average temperature of the configuration changes with the flattening (increasing e) of the boundary and also the loss of energy by radiation. As no subatomic energy generation is taken into account this investigation cannot, indeed, give an account of the change in the luminosity of a stellar body in course of its evolution.

If at any stage W_T be the kinetic, W_p the gravitational and W_i the heat energy of the configuration, we put

$$W = W_T + W_p + W_i \quad (4.1)$$

where

$$W_T = \int \frac{1}{2} \omega^2 r^2 \rho dv = \frac{1}{2} \omega^2 I \quad (4.2)$$

I being defined by

$$I = \frac{2}{3} Ma^2 \quad (4.3)$$

and represents the moment of inertia of the model;

$$W_p = -\frac{1}{2} \int V \rho dv \quad (4.4)$$

and

$$W_i = c_v \int T \cdot \rho dv \quad (4.5)$$

in which T stands for the absolute temperature, c_v denotes the specific heat at constant volume of the material and V , the Newtonian potential given by (1.13).

We shall calculate the total heat-content represented by (4.5) by assuming, as stated before, that the rotating material is a gas for which the *equation of state* is

$$p = \lambda \rho T \quad (4.6)$$

p and ρ being, of course, given by (1.6) and (1.5) respectively. Now, from (4.2), (4.3), (2.3), (3.4) we have

$$W_T = \frac{4}{35} \cdot \pi \rho_0 \cdot Ma^2 \cdot E(x). \quad (4.7)$$

From (4.4), (1.13), (2.7) and the following integrals,

$$\begin{aligned} A_0 &= \int_0^\infty \frac{d\theta}{(a^2 + \theta)(c^2 + \theta)} = \frac{2}{c} (x \cot^{-1} x) \\ \int r^2 \rho dv &= \frac{2}{7} Ma^2; \quad \int z^2 \rho dv = \frac{1}{7} Mc^2; \quad \int r^2 z^2 \rho dv = \frac{2}{63} Ma^2 c^2; \\ \int r^4 \rho dv &= \frac{8}{63} Ma^4; \quad \int z^4 \rho dv = \frac{1}{21} Mc^4; \quad \int \rho^2 dv = \frac{4}{7} \rho_0 M; \\ \int z^2 \rho^2 dv &= \frac{4}{63} \rho_0 Mc^2; \quad \int \rho^2 r^2 dv = \frac{8}{63} \rho_0 Ma^2 \end{aligned} \quad (4.8)$$

and (1.19), we have after some calculations,

$$W_p = -\frac{8}{21} \pi \rho_0 Ma^2 (x \cot^{-1} x). \quad (4.9)$$

Again, from (4.5) and (4.6), we have,

$$W_i = \frac{c_v}{\lambda} \left[(k\rho_0 + k') \int \rho^2 dv - (k\rho_0 \alpha + mk') \int \rho^2 r^2 dv - k\rho_0 \beta \int z^2 \rho^2 dv \right].$$

Hence, substituting the values from (4.8) we have

$$W_i = \frac{4}{7} \cdot \frac{c_v}{\lambda} \cdot \rho_0 \cdot Ma^2 \left[\frac{2}{3} \left(\frac{k\rho_0}{a^2} \right) + \frac{k'}{a^2} - \frac{2}{9} mk' \right]$$

Substituting from (2.7) and simplifying, we have

$$W_i = \frac{c_v}{\lambda} \cdot \frac{8}{21} \cdot \pi \rho_0 \cdot M a^2 x^2 (1 - x \cot^{-1} x) \quad (4.10)$$

Equations (4.7), (4.9), (4.10) can be written in the forms

$$W_T = \frac{3}{14} \cdot M^2 \cdot \frac{1}{a} E(x) \left(1 + \frac{1}{x^2}\right)^{\frac{1}{2}} \quad (4.11a)$$

$$W_p = -\frac{5}{7} \cdot M^2 \cdot \frac{1}{a} \cdot (x \cot^{-1} x) \left(1 + \frac{1}{x^2}\right)^{\frac{1}{2}} \quad (4.11b)$$

$$\text{and} \quad W_i = \frac{c_v}{\lambda} \cdot \frac{5}{7} \cdot M^2 \cdot \frac{1}{a} \cdot x^2 (1 - x \cot^{-1} x) \left(1 + \frac{1}{x^2}\right)^{\frac{1}{2}} \quad (4.11c)$$

Let us first consider the limiting case of the spherical configuration when $\epsilon \rightarrow 0$.

This can be divided into two sub-cases according as we consider a to be finite or infinite. If a be infinite, we have each of W_T , W_p , W_i vanishing, which represents the state of infinite diffusion. Hence, agreeing with our original assumption the total energy in this case just vanishes. If a is taken to have a finite value (as argued in Sec. 3) we have the following values ($x \rightarrow \infty$).

$$\lim_{x \rightarrow \infty} W_T = 0 \quad (4.12a)$$

$$\lim_{x \rightarrow \infty} W_p = -\frac{4}{7} \cdot M^2 \cdot \frac{1}{a} \quad (4.12b)$$

$$\lim_{x \rightarrow \infty} W_i = \frac{c_v}{\lambda} \cdot \frac{5}{21} \cdot M^2 \cdot \frac{1}{a} \quad (4.12c)$$

$$\text{Therefore} \quad \lim_{x \rightarrow \infty} W = -\frac{5}{7} \cdot M^2 \cdot \left(1 - \frac{1}{3} \frac{c_v}{\lambda}\right) \quad (4.13)$$

λ , of course, being $c_p - c_v$. We know that for a monatomic gas $c_v/\lambda = \frac{3}{2}$ and for a diatomic gas $c_v/\lambda = \frac{5}{2}$. Hence, for both cases W remains negative. In fact, in this case, W is negative for all values of $\gamma(c_p/c_v)$ greater than $\frac{4}{3}$.

Let us consider the other extreme, $\epsilon \rightarrow 1$. As in this case $x \rightarrow 0$, a tends to its minimum value given by (3.18). Hence,

$$\lim_{x \rightarrow 0} W_T = \frac{3}{14} \cdot M^2 \cdot \frac{1}{a_{min}} \cdot \left(\frac{7\pi}{8}\right) \quad (4.14a)$$

$$\lim_{x \rightarrow 0} W_p = -\frac{5}{7} \cdot M^2 \cdot \frac{1}{a_{min}} \cdot \left(\frac{\pi}{2}\right) \quad (4.14b)$$

$$\text{and} \quad \lim_{x \rightarrow 0} W_i \rightarrow 0 \quad (4.14c)$$

Thus

$$\lim_{x \rightarrow 0} W = -M^2 \cdot \frac{1}{a_{min}} \cdot \frac{19\pi}{112} \quad (4.15)$$

and hence negative. Thus the attainment of any of the above two configurations implies loss of energy by radiation.

The expression for the total energy in the general case is given by

$$W = -\frac{4}{3}\pi\rho_0 Ma^2 \left[\frac{2}{3}(x \cot^{-1} x) - \mu x^2(1 - x \cot^{-1} x) - \frac{1}{3}E(x) \right] \quad (4.16)$$

where we have put

$$\frac{2c_p}{3\lambda} = \mu. \quad (4.17)$$

If W gives a negative value for $\mu = \frac{2}{3}$ it will do so for $\mu = 1$.

We shall now show that W remains negative for all values of x . For this we put

$$W = -\frac{4}{3}\pi\rho_0 Ma^2 \cdot W_1(x) \quad (4.18)$$

where

$$W_1(x) = \frac{2}{3}(x \cot^{-1} x) - \mu x^2(1 - x \cot^{-1} x) - \frac{1}{3}E(x) \quad (4.19)$$

The values of $W_1(x)$ for $\mu = \frac{2}{3}$, are shown in Table 4. This shows clearly that W remains negative throughout, for both monatomic and diatomic gases.

From the general expressions for $W(x)$ and $W_1(x)$, given by (4.18) and (4.19) we can deduce an interesting conclusion regarding the dependence of stability of the configurations under consideration on the value of γ , the ratio of the specific heats of the constituent gas. It is well-known that a non-rotating (spherical) mass of gas in equilibrium, such as a polytrope for instance, has no thermodynamic stability unless γ exceeds $\frac{4}{3}$. Since W represents the total energy of the configuration, stability requires that it should not be positive. As a matter of fact we see in Table 4 that W is negative for $\mu = \frac{2}{3}$ ($\gamma = \frac{7}{5}$). The formula (4.19) offers the possibility of finding the minimum value of γ for which W remains negative for a given eccentricity. For this, we note that

$$c_p - c_v = \lambda \quad (4.19a)$$

from (4.6). Also from (4.17), we have

$$\gamma = 1 + \frac{2}{3\mu} \quad (4.19b)$$

where $\gamma \geq \frac{4}{3}$ when $\mu \leq 2$. Now, in (4.19), the term in μ being negative for all values of the eccentricity, if $W_1(x)$ vanishes for a certain value of μ it will be negative for all higher values of μ and for the same eccentricity. Hence the largest possible value of μ consistent with stability is given by

$$\mu = \frac{\frac{2}{3}(x \cot^{-1} x) - \frac{1}{3}E(x)}{x^2(1 - x \cot^{-1} x)}$$

or, from (4.19b)

$$\gamma - 1 = \frac{x^2(1 - x \cot^{-1} x)}{x \cot^{-1} x - 0.3E(x)} \quad (4.19c)$$

From this, we find that, when $x \rightarrow \infty$, $\gamma \rightarrow \frac{4}{3}$ and when $x \rightarrow 0$, $\gamma \rightarrow 1$. Further, for

$$\begin{array}{ccccccccc} x = & 2, & 1, & 0.7, & 0.4, & 0.3, & 0.2, & 0.1 & \\ \gamma = & 1.31, & 1.30, & 1.27, & 1.25, & 1.22, & 1.15, & 1.09 & \end{array} \quad (4.19d)$$

respectively, giving the lowest values of γ admissible, for they make $W_1(x)$ vanish.

This result shows that stability of this type is increased by rotation and consequent flattening of the body.

Now, we can show that

$$\frac{4}{7} \pi \rho_0 M a^2 = \frac{15}{14} M^2 \left(1 + \frac{1}{x^2}\right)^{\frac{1}{2}} \cdot \frac{1}{a}, \quad (4.20)$$

and

$$\frac{1}{a} = \left(\frac{2M}{7A}\right)^{\frac{1}{2}} \cdot (L')^{\frac{1}{2}} E(x) \left(1 + \frac{1}{x^2}\right)^{\frac{1}{2}},$$

hence, (4.18) can be written as

$$W = -D \cdot E(x) \left(1 + \frac{1}{x^2}\right) W_1(x) \quad (4.21)$$

where D is a positive constant, given by

$$D = \frac{5}{4} M^2 (2M/7A)^{\frac{1}{2}} (L')^{\frac{1}{2}}. \quad (4.21a)$$

Table 4 shows the variation of W with x and for $\mu = \frac{5}{3}$.

TABLE 4

| x | e | $15W_1(x)$ | $-(15W/D)$ |
|-----------------|-----------------|------------|----------------------|
| 2 | 0.447 | 1.713 | 0.2607 |
| 1 | 0.707 | 1.77 | 0.832 |
| 0.8 | 0.78 | 1.77 | 1.308 |
| 0.6 | 0.85 | 1.70 | 2.21 |
| 0.3 | 0.957 | 1.06 | 6.028 |
| 0.1 | 0.99 | 0.64 | 12.8 |
| $\rightarrow 0$ | $\rightarrow 1$ | | \rightarrow finite |

This table shows clearly that the total energy diminishes monotonically from the spherical state of finite or infinite radius to the limit of extreme flattening. We therefore notice that as the model contracts and becomes flatter at the same time it loses energy. This loss may be interpreted as loss due to radiation.

Sec 5. THE INTERNAL ENERGY AND THE AVERAGE TEMPERATURE

For a model starting from a state of infinite diffusion the internal energy or the heat energy given by (4.11c) is zero at the start. It vanishes again when $e \rightarrow 1$ (4.14c), and it is positive throughout. Hence it must attain a maximum for a certain value of the eccentricity in between. Now, from (4.11c) and (4.20) we have,

$$\frac{W}{H} = E(x)(1 - x \cot^{-1} x)(1 + x^2) \quad (5.1)$$

where H is a positive constant, given by

$$H = \frac{c_s}{\lambda} \cdot \frac{5}{7} \cdot M^2 \cdot \left(\frac{2M}{7A}\right)^{\frac{1}{2}} \cdot (L')^{\frac{1}{2}}. \quad (5.1a)$$

As $(1-x \cot^{-1}x)(1+x^2)$ increases with increasing x , from $\frac{1}{2}$ at $x \rightarrow \infty$ to 1 at $x \rightarrow 0$, monotonically, one expects the maximum of W_i/H to depend practically on that of $E(x)$ and so the internal energy will attain its maximum near about the same point where $\bar{\omega}^2/2\pi\rho$ does so. This conjecture is borne out by the calculations for the average temperature.

We define the average temperature \bar{T} by

$$\bar{T} = \int T dv / \int dv. \quad (5.2)$$

Substituting from (4.6) and (1.0) we get on simplification

$$\bar{T} = \frac{3}{4\lambda} \cdot \frac{M}{\pi c} \left[\frac{4}{21} \cdot \left(\frac{3k\rho_0}{a^2}\right) + \frac{1}{2} \left(\frac{2k'}{a^2}\right) - \frac{1}{7} \cdot 2mk' \right] \quad (5.3)$$

Hence, from (2.7), (3.3) and (3.15) we obtain,

$$\bar{T} = \frac{1}{\lambda} \cdot M \cdot \left(\frac{2M}{7A}\right)^{\frac{1}{2}} \cdot (L')^{\frac{1}{2}} \cdot E(x) \cdot (1+x^2) \cdot f(x)^{-\frac{1}{2}} \quad (5.4)$$

where

$$f(x) = \frac{38}{21} - \frac{2}{7} \cdot x^2 + \frac{2}{7} \cdot (x^2-6)(x \cot^{-1}x)$$

or putting,

$$\bar{T} = T_0 \cdot \theta(x) \quad (5.5)$$

we have

$$\theta(x) = E(x) \cdot (1+x^2) f(x) \quad (5.6)$$

and

$$T_0 = \frac{1}{\lambda} \cdot M \cdot \left(\frac{2M}{7A}\right)^{\frac{1}{2}} \cdot (L')^{\frac{1}{2}} \quad (5.6a)$$

For $x \rightarrow 0$, $\theta(x) \rightarrow 0$ and hence $T \rightarrow 0$. For $x \rightarrow \infty$, $\bar{T} \rightarrow \frac{1}{35} \cdot (1/\lambda) \cdot (M/a)$ or zero, according as a is finite or infinite [(5.4) and (2.9)].

The following table gives the values of $f(x)$ and $\theta(x)$ and shows how the average temperature of the configuration measured in units of T_0 varies with the flattening of the external boundary section.

It may be noted that the temperature attains a maximum at $e = 0.98$ or thereabout, practically for the same value of the eccentricity for which $\bar{\omega}^2/2\pi\rho$ attains its maximum value. The last part in the variation of the temperature and its ultimate vanishing may be looked upon as the inevitable consequence of the assumptions made regarding the boundary condition and the equation of state inside. With pressure always vanishing on the boundary the assumption of the ordinary gas law would imply a temperature vanishing in the limiting case when the eccentricity tends to unity leading to complete flattening.

TABLE 5

| x | e | $\frac{7}{2}f(x)$ | $\frac{7}{2}\theta(x)$ |
|-----------------|-----------------|---------------------|--|
| ∞ | 0 | | $\rightarrow 0 (a \rightarrow \infty)$ |
| 5 | 0.196 | 0.124 | 0.0522 |
| 2 | 0.447 | 0.477 | 0.2826 |
| 1 | 0.707 | 1.407 | 0.6618 |
| 0.8 | 0.78 | 1.852 | 0.8774 |
| 0.7 | 0.81 | 2.14 | 1.0193 |
| 0.6 | 0.85 | 2.488 | 1.1680 |
| 0.5 | 0.89 | 3.001 | 1.3807 |
| 0.4 | 0.92 | 3.394 | 1.4960 |
| 0.3 | 0.957 | 3.976 | 2.0322 |
| 0.2 | 0.98 | 4.688 | 1.5897 |
| 0.1 | 0.99 | 5.444 | 1.098 |
| $\rightarrow 0$ | $\rightarrow 1$ | $\rightarrow 6.333$ | $\rightarrow 0$ |

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A PROBLEM IN RECTILINEAR CONGRUENCES USING TENSOR CALCULUS

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The object of this paper is to study the curves on the surface of reference of a rectilinear congruence, which have the property that their rectifying planes at all their points contain the lines of the congruence through those points. The curves relating to the congruence of normals to the surface of reference are asymptotic lines on the surface of reference. It is to be noted in this connection that Springer (1945, 1947) has studied the curves for which the osculating planes at all their points contain the lines of the congruence.

1. Let the coordinates of a point P on the surface of reference S be given by $x^i = x^i(u^1, u^2)$, ($i = 1, 2, 3$) and the direction cosines of the ray of the congruence through x^i by $\lambda^i = \lambda^i(u^1, u^2)$, ($i = 1, 2, 3$), then

$$\lambda^i \lambda^i = 1 \quad (1.1)$$

and

$$\lambda^i = p^\alpha x^i_{,\alpha} + q X^i \quad (1.2)$$

where p^α are the contravariant components of a vector in the surface at P ; $x^i_{,\alpha}$, ($\alpha = 1, 2$)* are the direction numbers and denote covariant differentiation of x^i with regard to u^α based on the first fundamental tensor

$$g_{\alpha\beta} = x^i_{,\alpha} \cdot x^i_{,\beta} \quad (1.3)$$

of the surface S ; and q is a positive scalar function.

From the equation (1.2), we get using (1.3)

$$\lambda^i \cdot x^i_{,\alpha} = p^\beta x^i_{,\beta} \cdot x^i_{,\alpha} = p^\beta g_{\beta\alpha} = p_\alpha. \quad (1.4)$$

If θ is the angle between the normal to the surface at P and the line λ of the congruence at P , it follows from (1.2) that

$$\cos \theta = \lambda^i \cdot X^i = q. \quad (1.5)$$

The direction cosines of the principal normal to the curve $C: x^i = x^i(s)$ on S at P are proportional to $x^{i''}$, where dashes denote differentiations with regard to s , the arc length of the curve. Therefore the differential equation of the rectifying plane to the curve at P is given by

$$\sum_i (\xi^i - x^i) x^{i''} = 0.$$

Now $x^{i'} = x^i_{,\alpha} u'^\alpha$ and

$$x^{i''} = x^i_{,\alpha} u''^\alpha + \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} u'^\alpha u'^\beta. \quad (1.7)$$

* In what follows Latin indices take the values (1, 2, 3) and Greek indices the values (1, 2).

But by Gauss's equations of S

$$\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} = \left\{ \begin{matrix} \gamma \\ \alpha \beta \end{matrix} \right\} x^i_{,\gamma} + d_{\alpha\beta} X^i \quad (1.8)$$

where $\left\{ \begin{matrix} \gamma \\ \alpha \beta \end{matrix} \right\}$ are the Christoffel symbols of the second kind and $d_{\alpha\beta}$ is the second fundamental tensor of the surface S .

By virtue of (1.7) and (1.8), the equation (1.6) assumes the form

$$\sum_i (\xi^i - x^i) (\rho^\gamma x^i_{,\gamma} + k_n X^i) = 0 \quad (1.9)$$

where ρ^γ are the components of the curvature vector of the curve C at P (Eisenhart, 1940, p. 187) and k_n is the normal curvature of the surface in the direction u'^α ($\alpha = 1, 2$).

Rectifying plane of the curve C at P , on the surface of reference is given by (1.9). If it contains the line λ of the congruence, the coordinates

$$\xi^i = x^i + t\lambda^i$$

must satisfy the equation (1.9).

Hence, by use of (1.4) and (1.5), we have

$$\rho^\gamma p_\gamma + q k_n = 0. \quad (1.10)$$

This is an equation of the second order, of the curves on S , through P , which are such that their rectifying planes at all the points contain the lines of the congruence through those points. There are, therefore, ∞^2 such curves on the surface.

If the congruence is formed by tangents to the curves of any one parameter family on the surface of reference

$$q = 0, \quad (1.11)$$

and

$$p_\alpha \equiv \lambda^i x^i_{,\alpha} = x^i_{,\alpha} x^i_{,\beta} u'^\beta = g_{\alpha\beta} u'^\beta. \quad (1.12)$$

In consequence of (1.11) and (1.12), the equation (1.10) assumes the form

$$\rho^\alpha g_{\alpha\beta} u'^\beta = 0$$

which is true for any curve on the surface.

Hence if the congruence is formed by tangents to a one-parameter family of curves on the surface of reference S , then the curves C such that the rectifying plane at any point contains the ray of the congruence through that point are the same as given one-parameter family of curves.

For a congruence formed by normals to the surface,

$$p_\alpha = 0$$

and the equation (1.10) becomes, $k_n = 0$, since $q \neq 0$, that is, if the congruence is formed by normals to the surface of reference, these curves become asymptotic lines.

Conversely, if these curves are asymptotic lines, i.e. $k_n = 0$, then from (1.10) we see that for these curves $\rho^\gamma p_\gamma = 0$ which is satisfied if the congruence is formed by normals to the surface of reference. The other possibility viz., that $\rho^\gamma = 0$ is to be rejected as then the asymptotic lines become geodesics, in which case the curves

become straight lines and the surface of reference becomes a ruled surface. The third possibility may occur, when neither p_γ nor ρ^γ is zero, but the curves satisfy the differential equation $\rho^1 p_1 + \rho^2 p_2 = 0$

2. Let us now consider the curvature of these curves. Suppose the direction cosines of the tangent, principal normal and binormal to C at P are $\alpha^i, \beta^i, \gamma^i$. The binormal to C lies in the plane of the lines with directions λ^i and $\alpha^i (= dx^i/ds)$, hence

$$\gamma^i = a \frac{dx^i}{ds} + b \lambda^i \quad (2.1)$$

where the functions a and b are to be determined.

Multiplying (2.1) by dx^i/ds and summing for i we get

$$a = -\cot \varphi.$$

Similarly taking the scalar product of both sides with λ^i we get

$$b = \operatorname{cosec} \varphi.$$

where φ , the angle between the tangent to the curve C through P on the surface and the line of the congruence through P with direction λ^i , is given by

$$\cos \varphi = \lambda^i \frac{dx^i}{ds} = \lambda^i x^i_{,a} u'^a = p_a u'^a. \quad (2.2)$$

Hence the equation (2.1) assumes the form

$$\gamma^i = \operatorname{cosec} \varphi \left(\lambda^i - \cos \varphi \frac{dx^i}{ds} \right) \quad (2.3)$$

By Frenet's formula

$$k = \beta^i \frac{d\alpha^i}{ds} \quad (2.4)$$

where k is the curvature of the curve C at x^i . As

$$\beta^i = \gamma^i \times \alpha^i \quad (2.5)$$

and

$$\frac{d\alpha^i}{ds} = \frac{d^2 x^i}{ds^2} = \rho^\gamma x^i_{,\gamma} + k_n X^i \quad (2.6)$$

the equation (2.4) assumes the form,

$$k = \operatorname{cosec} \varphi \left(\rho^\gamma x^i_{,\gamma} + k_n X^i - \lambda^i - \cos \varphi \frac{dx^i}{ds} \right) \frac{dx^i}{ds} = \operatorname{cosec} \varphi \left(\rho^\gamma x^i_{,\gamma} + k_n X^i - \lambda^i \frac{dx^i}{ds} \right) \quad (2.7)$$

Use of (1.2) in (2.7) yields,

$$k = \operatorname{cosec} \varphi (\rho^\gamma x^i_{,\gamma} + k_n X^i - p^a x^i_{,a} + q X^i - x^i_{,\beta}) u'^\beta$$

or

$$k = \operatorname{cosec} \varphi (k_n p^a - q \rho^a) e_{a\beta} u'^\beta \quad (2.8)$$

where

$$e_{a\beta} = (X^i - x^i_{,a} - x^i_{,\beta})$$

If k_u is the union curvature of the curve, then (Springer, 1945)

$$k_u = e_{a\beta} (k_n l^a - \rho^a) u'^\beta \quad (2.9)$$

where $l^2 = p^2/q$. Use of (2.9) in (2.8) yields,

$$k = \operatorname{cosec} \varphi \cdot \cos \theta \cdot k_u. \quad (2.10)$$

Another expression for the curvature can be found as follows:

$$\frac{d\alpha^i}{ds} = k\beta^i$$

By virtue of (2.3), (2.5) and (2.6), this equation becomes,

$$\rho^\gamma x^i_{,\gamma} + k_n X^i = k \operatorname{cosec} \varphi \cdot \left(\lambda^i - \cos \varphi \frac{dx^i}{ds} \right) \times \frac{dx^i}{ds} = k \operatorname{cosec} \varphi \cdot \lambda^i \times \frac{dx^i}{ds} \quad (2.11)$$

Multiplying (2.11) by X^i and summing for 'i' we get,

$$k_n = k \operatorname{cosec} \varphi \cdot \left(X^i \lambda^i - \frac{dx^i}{ds} \right) = k \operatorname{cosec} \varphi \cdot (X^i p^a x^i_{,a} + q X^i x^i_{, \beta} u'^\beta) = k \operatorname{cosec} \varphi \cdot e_{a\beta} p^a u'^\beta,$$

whence,

$$k = \frac{k_n \sin \varphi}{e_{a\beta} p^a u'^\beta}. \quad (2.12)$$

From the two expressions (2.8) and (2.12) for k , we get the curvature of the normal section of the surface for the direction of the curve.

$$k_n = \frac{q k_p e_{a\beta} p^a u'^\beta}{(e_{a\beta} p^a u'^\beta)^2 - \sin^2 \varphi} \quad (2.18)$$

where k_p , the geodesic curvature of the curve is given by

$$k_p = e_{a\beta} \rho^a u'^\beta.$$

Some other expressions for the curvature of these curves can be obtained from the equation (2.11).

Multiplying the equation (2.11) by λ^i and summing for 'i' we get,

$$\rho^\gamma p_\gamma + k_n q = 0 \quad (2.14)$$

which is the equation of the curve (cf. 1.10).

3. We shall now find the torsion of these curves. By Frenet's formula, the torsion of a curve is given by,

$$\tau = \beta^i \frac{d\gamma^i}{ds} \quad (3.1)$$

Use of (2.3) and (2.5) in (3.1) yields,

$$\begin{aligned} \tau = \operatorname{cosec}^2 \varphi \cdot & \left(\frac{d\lambda^i}{ds} - \cos \varphi \frac{d^2 x^i}{ds^2} + \frac{dx^i}{ds} \sin \varphi \frac{d\varphi}{ds} - \cot \varphi \frac{d\varphi}{ds} \lambda^i \right. \\ & \left. + \cos \varphi \cot \varphi \frac{d\varphi}{ds} \frac{dx^i}{ds} - \lambda^i - \cos \varphi \frac{dx^i}{ds} \frac{dx^i}{ds} \right). \end{aligned}$$

On dropping vanishing determinants we get,

$$\tau = \operatorname{cosec}^2 \varphi \left[\left(\frac{d\lambda^i}{ds} - \lambda^i \frac{dx^i}{ds} \right) - \cos \varphi \left(\lambda^i \frac{dx^i}{ds} - \frac{d^2 x^i}{ds^2} \right) \right] \quad (3.2)$$

By virtue of (1.2), (2.2) and the fact that (Ram Behari and Mishra, 1949)

$$\lambda^i_{,a} = \mu^i_a x^i_{,\gamma} + v_a X^i$$

we get,

$$\tau = \operatorname{cosec}^2 \varphi \cdot e_{ra} [(p^r v_\beta - q \mu^r_\beta) + p_\beta (k_n p^r - q \rho^r)] u'^a u'^\beta. \quad (3.3)$$

Use of (2.9) in (3.3) yields,

$$\tau = \operatorname{cosec}^2 \varphi \cdot [\cos \theta \cos \varphi \cdot k_u + e_{ra} (p^r v_\beta - q \mu^r_\beta) u'^a u'^\beta]. \quad (3.4)$$

For a congruence formed by normals to the surface of reference,

$$\tau = e_{ar} \mu^r_\beta u'^a u'^\beta = e_{ra} d_{\beta\sigma} g^{\sigma r} u'^a u'^\beta = \tau_g.$$

Thus *the torsion of an asymptotic line is its geodesic torsion* (a known result).

Also

$$\tau \beta^i = \frac{d\gamma^i}{ds}. \quad (3.5)$$

Using (2.3) and (2.5) in (3.5) we get,

$$-\cot \varphi \frac{d\varphi}{ds} \left(\lambda^i - \cos \varphi \frac{dx^i}{ds} \right) + \frac{d\lambda^i}{ds} + \sin \varphi \frac{d\varphi}{ds} \cdot \frac{dx^i}{ds} - \cos \varphi \frac{d^2 x^i}{ds^2} = \tau \left(\lambda^i - \cos \varphi \frac{dx^i}{ds} \right) \times \frac{dx^i}{ds} \quad (3.6)$$

Multiplying both sides by λ^i and summing for 'i' we get

$$\rho^r p_r + q k_n = 0$$

which is the equation of the curve (cf. 1.10).

Other expressions for the torsion of the curve may be obtained from the equation (3.6) by multiplying it by $d\lambda^i/ds$, $d^2 x^i/ds^2$ etc. and summing for 'i'.

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HOMOLOGY GROUPS OF A RING—I

By

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1. INTRODUCTION

Let S be a compact Hausdorff space and $C(S)$ denote the ring of all continuous real functions defined on S . According to a result of I. Gelfand and G. E. Shilov (1941), the algebraic structure of the ring $C(S)$ determines the space S up to a homeomorphism. This exciting result raised a natural problem to characterize all the topological properties of the space S in terms of the purely algebraic structure of the ring $C(S)$.

The present paper, which is the first of a projected series, is concerned with the characterization of the homology invariants of the space S by means of those of the ring $C(S)$.

For a given abstract ring R , homology and cohomology theories can be obviously defined in various ways. A cohomology theory of rings was introduced by G. P. Hochschild (1945) and others; however, this was not designed to fulfil our purpose. In the present paper, homology groups of a ring R are defined upon the pattern of the Alexander-Kolmogoroff-Lefschetz homology groups of a topological space. It is proved that, if the ring R is the ring $C(S)$ of all continuous real functions defined on a compactum S , then the homology groups of R over any topological coefficient group G are isomorphic and homeomorphic with the corresponding topologized Alexander-Kolmogoroff-Lefschetz homology groups of the compactum S .

2. PRELIMINARY ALGEBRAIC CONCEPTS

Throughout the paper, let R denote a given ring. By a *proper* right ideal of R , we shall understand a right ideal of R which is not the whole ring R . A proper right ideal M of R is said to be *maximal* if, for every proper right ideal I of R , $I \supset M$ implies $I \subset M$. The intersection of all maximal right ideals of R will be called the *radical* of R , denoted by $\rho(R)$. If the ring R has a left unity element, then the above definition is equivalent with that of N. Jacobson (1945).

A finite set (x_0, \dots, x_n) of elements of R is said to be *large* if it is contained in no maximal right ideal of R .

A pair (u, v) of elements of R is said to be *segregated* if, for each element

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$x \in R \setminus \rho(R)$, there is a maximal right ideal M of R not containing x and containing at most one of the elements u and v .

3. THE GROUP OF n -CHAINS

$R^{(n+1)}$ will denote the $(n+1)$ -fold cartesian product of R with itself; that is,

$$R^{(n+1)} = \{(x_0, \dots, x_n) \mid x_i \in R, i=0, \dots, n\}.$$

Let G denote a given topological abelian group, called the coefficient group.

The group $C_n(R, G)$ of n -chains of R over G is defined to be the set of all functions $\phi: R^{(n+1)} \rightarrow G$ (with functional addition as the group operation), satisfying the following conditions:

(3.1). ϕ is alternate; i.e., if (i_0, \dots, i_n) is a permutation of the $n+1$ integers $(0, \dots, n)$ then

$$\phi(x_{i_0}, \dots, x_{i_n}) = \text{sgn} \begin{pmatrix} 0, \dots, n \\ i_0, \dots, i_n \end{pmatrix} \phi(x_0, \dots, x_n),$$

where $\text{sgn}(P)$ is 1 or -1 according as the permutation P is even or odd.

(3.2). ϕ is largely zero; i.e., if (x_0, \dots, x_n) is a large set of R , then $\phi(x_0, \dots, x_n) = 0$.

(3.3). ϕ is quasi-additive; i.e., if for a certain fixed index i , the element $x_i = uv$, where (u, v) is a segregated pair, then

$$\phi(x_0, \dots, x_n) = \phi(x_0, \dots, x_{i-1}, u, x_{i+1}, \dots, x_n) + \phi(x_0, \dots, x_{i-1}, v, x_{i+1}, \dots, x_n).$$

The verification that the set $C_n(R, G)$ form an abelian group with functional addition as the group operation is immediate.

A topology can be introduced into the group $C_n(R, G)$ as follows. Let U be a neighborhood of the zero element in G and A be a finite set of elements of R . Let us denote by $V(A, U)$ the set of all $\phi \in C_n(R, G)$ such that $\phi(x_0, \dots, x_n) \in U$ whenever $x_i \in A$ for each $i = 0, \dots, n$. The family $\{V(A, U)\}$ of all such sets is used as a system of neighborhoods of the zero element of $C_n(R, G)$.

4. SOME ELEMENTARY PROPERTIES OF CHAINS

An element $x \in R$ is said to be large if it is contained in no maximal right ideal of R . The following statement is an obvious special case of (3.2).

(4.1). If some element $x_i \in R$ is large, then for each chain $\phi \in C_n(R, G)$ we have $\phi(x_0, \dots, x_i, \dots, x_n) = 0$.

The above assertion implies the following consequences.

(4.2). If the ring R has a left unity element e , then $\phi(x_0, \dots, x_n) = 0$ if some element $x_i = e$.

(4.3). If the ring R has a unity element e , then $\phi(x_0, \dots, x_n) = 0$ if some element x_i has a right inverse.

5. THE BOUNDARY OPERATION

To each n -chain $\phi \in C_n(R, G)$, ($n > 0$), let us define the boundary $\partial\phi$ of ϕ to be the function $\partial\phi: R^{(n)} \rightarrow G$ as follows:

$$\partial\phi(x_0, \dots, x_{n-1}) = \phi(\theta, x_0, \dots, x_{n-1})$$

for an arbitrary $(x_0, \dots, x_{n-1}) \in R^{(n)}$, where θ denotes the zero element of R . The conditions (3.1)–(3.3) are obviously satisfied by $\partial\phi$; hence $\partial\phi \in C_{n-1}(R, G)$. Further, one can easily verify the following assertion.

(5.1). *The correspondence $\phi \rightarrow \partial\phi$ is a continuous homomorphism (into).*

$$\partial : C_n(R, G) \rightarrow C_{n-1}(R, G).$$

The condition (3.1) for ϕ implies immediately the following statement.

(5.2). $\partial\partial = 0$; i.e., for each chain $\phi \in C_n(R, G)$, ($n \geq 2$), $\partial\partial\phi$ is the zero element of $C_{n-2}(R, G)$.

6 THE HOMOLOGY GROUPS OF R

If $n > 0$, an n -chain $\phi \in C_n(R, G)$ is called an n -cycle of R over G whenever $\partial\phi = 0$. The set of n -cycles of R over G will be denoted by $Z_n(R, G)$. Being the kernel of the continuous homomorphism ∂ , $Z_n(R, G)$ is a closed subgroup of $C_n(R, G)$. We define $Z_0(R, G) = C_0(R, G)$.

An n -chain $\phi \in C_n(R, G)$ is called an n -boundary of R over G , if there exists an $(n+1)$ -chain $\psi \in C_{n+1}(R, G)$ such that $\phi = \partial\psi$. The set of n -boundaries of R over G will be denoted by $B_n(R, G)$. Being the image of the homomorphism ∂ , $B_n(R, G)$ is a subgroup of $C_n(R, G)$, but not necessarily closed.

The following inclusion is an immediate consequence of (5.2) and the definition of $Z_0(R, G)$.

$$(6.1) \quad B_n(R, G) \subset Z_n(R, G).$$

Let $\bar{B}_n(R, G)$ denote the closure of $B_n(R, G)$ in $C_n(R, G)$. Then $\bar{B}_n(R, G)$ is a closed subgroup of $C_n(R, G)$ contained in $Z_n(R, G)$ because of the closedness of the latter, i.e.,

$$(6.2) \quad \bar{B}_n(R, G) \subset Z_n(R, G).$$

The quotient topological group

$$H_n(R, G) = Z_n(R, G) / B_n(R, G)$$

is defined to be the n -dimensional homology group of R over G .

7. HOMOLOGY GROUPS OF A TOPOLOGICAL SPACE

In the present paragraph, we shall recall briefly the definition of the Alexander-Kolmogoroff homology groups as formulated by S. Lefschetz (1942, p. 282).

Let S be a topological space. Separation axioms are not assumed. Let L denote the set of all closed sets of S . $L^{(n+1)}$ will denote the $(n+1)$ -fold cartesian product of L with itself; that is,

$$L^{(n+1)} = \{(F_0, \dots, F_n) \mid F_i \in L, i = 0, \dots, n\}.$$

Let G denote a given topological Abelian group called the coefficient group.

The group $C_n(S, G)$ of n -chains of S over G is defined to be the set of functions $\xi : L^{(n+1)} \rightarrow G$ (with functional addition as the group operation), satisfying the following conditions:

(7.1). ξ is alternate.

(7.2). $\xi(F_0, \dots, F_n) = 0$ if the intersection $F_0 \cap \dots \cap F_n$ is vacuous.

(7.3). ξ is quasi-additive; i.e., if for a certain fixed index i , the closed set $F_i = P \cup Q$, where P, Q are closed sets of S having no common interior point, then

$$\xi(F_0, \dots, F_n) = \xi(F_0, \dots, F_{i-1}, P, F_{i+1}, \dots, F_n) + \xi(F_0, \dots, F_{i-1}, Q, F_{i+1}, \dots, F_n).$$

The topology of $C_n(S, G)$ is defined as follows. Let U be a neighborhood of the zero element in G and a be a finite set of L . Let $W(a, U)$ denote the set of all $\xi \in C_n(S, G)$ such that $\xi(F_0, \dots, F_n) \in U$ whenever $F_i \in a$ for each $i = 0, \dots, n$. The family $\{W(a, U)\}$ of all such sets is used as a system of neighborhoods of the zero element of $C_n(S, G)$.

For each $n > 0$, the boundary $\partial\xi$ of an n -chain $\xi \in C_n(S, G)$ is defined to be the function $\partial\xi: L^{(n)} \rightarrow G$ as follows:

$$\partial\xi(F_0, \dots, F_{n-1}) = \xi(S, F_0, \dots, F_{n-1})$$

for an arbitrary $(F_0, \dots, F_{n-1}) \in L^{(n)}$. It is easily verified that $\partial\xi \in C_{n-1}(S, G)$ and that the correspondence $\xi \rightarrow \partial\xi$ is a continuous homomorphism:

$$\partial: C_n(S, G) \rightarrow C_{n-1}(S, G).$$

The condition (7.1) for ξ implies that $\partial\partial = 0$, i.e., for each n -chain $\xi \in C_n(S, G)$, ($n \geq 2$), $\partial\partial\xi$ is the zero element of $C_{n-2}(S, G)$.

For each $n > 0$, let $Z_n(S, G)$ denote the kernel of the homomorphism ∂ in $C_n(S, G)$. We also define $Z_0(S, G) = C_0(S, G)$. For each $n \geq 0$, let $B_n(S, G) = \partial C_{n+1}(S, G)$. Since $\partial\partial = 0$, we have

$$B_n(S, G) \subset \bar{B}_n(S, G) \subset Z_n(S, G),$$

where $\bar{B}_n(S, G)$ denotes the closure of $B_n(S, G)$ in $C_n(S, G)$. The quotient topological group

$$H_n(S, G) = Z_n(S, G) / \bar{B}_n(S, G)$$

is called the n -dimensional Alexander-Kolmogoroff-Lefschetz homology group of S over G , or simply the n -dimensional AKL homology group of S over G .

The following theorem was proved by S. Lefschetz (1942, p. 285).

(7.4). The AKL homology groups of a normal Hausdorff space S over a coefficient group G which is compact or a field are isomorphic and homeomorphic with the corresponding Čech homology groups based on the finite open coverings.

8. THE MAIN THEOREM

Throughout the remaining of the paper, S will denote a compactum, i.e., a compact metric space. Let $R = C(S)$ be the ring of all continuous real functions defined on S . In the sequel, we shall be concerned with the proof of the following theorem.

(8.1). **Theorem.** For any topological Abelian group G , the n -dimensional AKL homology group $H_n(S, G)$ is isomorphic and homeomorphic with the n -dimensional homology group $H_n(R, G)$ of the ring $R = C(S)$.

As a consequence of (7.4) and (8.1), we state the following

(8.2). **Corollary.** *If the coefficient group G is compact or a field, then the homology groups $H_n(R, G)$ of the ring $R = C(S)$ are isomorphic and homeomorphic with the corresponding Čech homology groups based on the finite open coverings.*

9. A FEW AUXILIARY ASSERTIONS

For each element x of the ring $R = C(S)$, let us denote by Z_x the *zero set* of x , i.e., the closed set of S defined by

$$Z_x = \{p \in S \mid x(p) = 0\}.$$

(9.1). *The collection $\{Z_x \mid x \in R\}$ of all zero sets contains all closed sets of S .*

Proofs. Let F be an arbitrarily given closed set. Define a real valued function x on S by taking

$$x(p) = d(p, F), \quad (p \in S),$$

where $d(p, F)$ denotes the distance from p to F . Clearly $x \in C(S)$ and $F = Z_x$. Q.E.D.

For each point $a \in S$, let us denote by M_a the set of elements of $C(S)$ defined by

$$M_a = \{x \in C(S) \mid x(a) = 0\}.$$

According to Gelfand and Shilov (1941), M_a is a maximal ideal of $C(S)$ and the family $\{M_a \mid a \in S\}$ contains all maximal ideals of $C(S)$.

(9.2). *A finite set (x_0, \dots, x_n) of $C(S)$ is large if and only if the intersection $\bigcap_{i=0}^n Z_{x_i}$ is vacuous.*

Proof. By definition, (x_0, \dots, x_n) is large if and only if there exists no maximal ideal of $C(S)$ which contains the finite set (x_0, \dots, x_n) . Since the family $\{M_a \mid a \in S\}$ contains all maximal ideals of $C(S)$, we conclude that (x_0, \dots, x_n) is large if and only if there exists no point $a \in S$ such that $x_i(a) = 0$ for each $i = 0, \dots, n$. The latter condition is equivalent with the condition that $\bigcap_{i=0}^n Z_{x_i}$ is empty.

(9.3). *A pair (u, v) of elements of $C(S)$ is segregated if and only if Z_u and Z_v have no common interior point.*

Proof. Necessity. Assume that (u, v) be segregated. Let W be an arbitrary non-vacuous open set of S and $F = S \setminus W$. According to (9.1), there exists an element x of $C(S)$ such that $Z_x = F$. Since W is non-vacuous and $a \in W$ implies x not $\in M_a$, x is not in the radical $\rho(C(S))$ of $C(S)$. Hence, by the definition of segregated pairs, there exists a maximal ideal M of $C(S)$ which does not contain x and which contains at most one of the elements u and v . Since the family $\{M_a \mid a \in S\}$ contains all maximal ideals of $C(S)$, there is a point $p \in S$ such that $M_p = M$. x not $\in M_p$ implies p not $\in Z_x = F$. Hence $p \in W$. Since M_p contains at most one of the elements u and v , p is contained in at most one of the sets Z_u and Z_v . Since W is an arbitrary non-vacuous open set, Z_u and Z_v can have no common interior point.

Sufficiency. Suppose that Z_u and Z_v have no common interior point. Let x be an arbitrary element of $C(S)$ not in the radical $\rho(C(S))$ of $C(S)$. The open set $W = S \setminus Z_x$ is non-vacuous. Since Z_u and Z_v have no common interior point, there exists a point $p \in W$ which belongs to at most one of the sets Z_u and Z_v . Then the

maximal ideal M_p of $C(S)$ does not contain x and contains at most one of the elements u and v . This prove that (u, v) is segregated and completes the proof.

10. THE HOMEOMORPHIC ISOMORPHISM κ

For an arbitrary n -chain $\xi \in C_n(S, G)$ of the compactum S over G , let us define a function $\phi = \kappa(\xi): R^{(n+1)} \rightarrow G$ by taking

$$\phi(x_0, \dots, x_n) = \xi(Z_{x_0}, \dots, Z_{x_n}).$$

By means of (9.2) and (9.3), the conditions (3.1)–(3.3) are easy consequences of the corresponding conditions (7.1)–(7.8). Hence $\phi \in C_n(C(S), G)$, i.e., ϕ is an n -chain of the ring $C(S)$ over G . Further, it is also clear that the correspondence $\xi \rightarrow \kappa(\xi)$ is a group-theoretic homomorphism of $C_n(S, G)$ into $C_n(C(S), G)$. We shall prove the following theorem.

(10.1). *The homomorphism κ maps $C_n(S, G)$ isomorphically and homeomorphically onto $C_n(C(S), G)$ and commutes with the boundary homomorphism ∂ , i.e., $\partial\kappa = \kappa\partial$ in the following diagram:*

$$\begin{array}{ccc} C_n(S, G) & \xrightarrow{\kappa} & C_n(C(S), G) \\ \downarrow \partial & & \downarrow \partial \\ C_{n-1}(S, G) & \xrightarrow{\kappa} & C_{n-1}(C(S), G). \end{array}$$

Proof. First, let us prove that κ is onto. Let $\phi \in C_n(C(S), G)$ be an arbitrary n -chain of the ring $C(S)$ over G . We shall define an n -chain $\xi \in C_n(S, G)$ as follows: Let F_0, \dots, F_n be arbitrary $n+1$ closed sets of S . Since the family $\{Z_x | x \in C(S)\}$ contains all closed sets of S , there exist elements x_0, \dots, x_n of $C(S)$ such that

$$F_i = Z_{x_i} \quad (i = 0, \dots, n).$$

We define the desired n -chain $\xi \in C_n(S, G)$ by taking

$$\xi(F_0, \dots, F_n) = \phi(x_0, \dots, x_n).$$

To justify this definition, let us show that $\xi(F_0, \dots, F_n)$ does not depend on the choice of the elements x_0, \dots, x_n . Suppose, for a fixed index i , we choose an element $y_i \in C(S)$ instead of x_i such that $F_i = Z_{y_i}$. Let

$$E_i = \text{Cl}(S \setminus F_i).$$

Since E_i is closed set of S , there exists an element $z_i \in C(S)$ such that $E_i = Z_{z_i}$. Since F_i and E_i have no common interior point, it follows from (9.3) that both the pairs (x_i, z_i) and (y_i, z_i) are segregated. Further, as a consequence of $S = F_i \cup E_i$, we obtain the relation

$$x_i z_i = \theta = y_i z_i,$$

where θ denotes the zero element of $C(S)$. By means of the condition (3.8), we deduce that both $\phi(x_0, \dots, x_i, \dots, x_n)$ and $\phi(x_0, \dots, y_i, \dots, x_n)$ are equal to

$$\phi(x_0, \dots, \theta, \dots, x_n) = \phi(x_0, \dots, z_i, \dots, x_n).$$

Hence we have

$$\phi(x_0, \dots, x_i, \dots, x_n) = \phi(x_0, \dots, y_i, \dots, x_n).$$

This proves that $\xi(F_0, \dots, F_n)$ depends only on the n -chain ϕ and the closed sets F_0, \dots, F_n . By the aid of (9.2) and (9.8), the conditions (7.1)—(7.3) for the functions ξ are easy consequences of the conditions (3.1)—(3.3) for the n -chain ϕ . Hence ξ is an n -chain of S over G , uniquely determined by ϕ . By the above construction of ξ and the uniqueness argument, it can be easily seen that $\phi = \kappa(\xi)$. Hence the homomorphism is onto.

Next, let us prove that κ is an isomorphism. Assume that $\xi \in C_n(S, G)$ be such that $\kappa(\xi) = 0$. For any $n+1$ closed sets F_0, \dots, F_n of S , let us choose $n+1$ elements x_0, \dots, x_n of $C(S)$ such that $F_i = Z_{x_i}$ ($i = 0, \dots, n$). Then, by definition of the homomorphism κ , we have

$$\xi(F_0, \dots, F_n) = \kappa(\xi)(x_0, \dots, x_n) = 0.$$

Hence $\xi = 0$ and κ is an isomorphism.

Now let us prove that κ is topological. Let U be an arbitrary neighborhood of the zero element in G . Let A be any finite set of elements of $C(S)$, and let

$$a = \{Z_x | x \in A\} \subset L.$$

Let $V = V(A, U)$ and $W = W(a, U)$ be the neighborhoods defined respectively in §3 and §7. According to the definition of κ , we have $\kappa^{-1}(V) = W$. Hence κ is continuous. On the other hand, let a^* be any finite set of closed sets of S . For each closed set $F \in a^*$, choose an element $x_F \in C(S)$ such that $Z_{x_F} = F$. Let $a^* = \{x_F | F \in a^*\}$. Then A^* is a finite set of $C(S)$. Let $V^* = V(A^*, U)$ and $W^* = W(a^*, U)$. By the aid of the argument used in the proof that κ is onto, one can easily see that $\kappa(W^*) = V^*$. Hence κ^{-1} is also continuous. This completes the proof that κ is a homeomorphic isomorphism of $C_n(S, G)$ onto $(C_n(C(S), G))$.

To prove the commutativity of κ and ∂ , let us observe for a given n -chain $\xi \in C_n(S, G)$, ($n > 0$):

$$\partial \kappa \xi(x_0, \dots, x_{n-1}) = \kappa \xi(\partial, x_0, \dots, x_{n-1}) = \xi(S, Z_{x_0}, \dots, Z_{x_{n-1}});$$

$$\kappa \partial \xi(x_0, \dots, x_{n-1}) = \partial \xi(Z_{x_0}, \dots, Z_{x_{n-1}}) = \xi(S, Z_{x_0}, \dots, Z_{x_{n-1}}).$$

Hence $\partial \kappa = \kappa \partial$. This completes the proof of (10.1).

(10.2). *The homeomorphic isomorphism*

$$\kappa : C_n(S, G) \rightarrow C_n(C(S), G)$$

maps $Z_n(S, G)$ onto $Z_n(C(S), G)$ and $B_n(S, G)$ onto $B_n(C(S), G)$. Therefore, κ induces a homeomorphic isomorphism (onto):

$$\kappa^* : H_n(S, G) \rightarrow H_n(C(S), G).$$

Proof. Let $\xi \in Z_n(S, G)$, then $\partial \kappa \xi = \kappa \partial \xi = 0$. Hence $\kappa \xi \in Z_n(C(S), G)$. Conversely, suppose $\xi \in C_n(S, G)$ be such that $\kappa \xi \in Z_n(C(S), G)$. Then, $\kappa \partial \xi = \partial \kappa \xi = 0$. Since κ is an isomorphism, we have $\partial \xi = 0$. Hence $\xi \in Z_n(S, G)$. This proves that

$$\kappa(Z_n(S, G)) = Z_n(C(S), G).$$

Next, let $\xi \in B_n(S, G)$. Then there exists an $(n+1)$ -chain $\eta \in C_{n+1}(S, G)$ such that $\xi = \partial\eta$. Hence

$$\kappa\xi = \kappa\partial\eta = \partial\kappa\eta \in B_n(C(S), G).$$

On the other hand, let $\xi \in C_n(S, G)$ be such that $k\xi \in B_n(C(S), G)$. There exists an $(n+1)$ -chain $\psi \in C_{n+1}(C(S), G)$, with $k\xi = \partial\psi$. Since κ is an isomorphism onto, $\partial\kappa = \kappa\partial$ implies $\kappa^{-1}\partial = \partial\kappa^{-1}$. Then $\xi = \kappa^{-1}\partial\psi = \partial\kappa^{-1}\psi$. Hence $\xi \in B_n(S, G)$. This proves that

$$\kappa(B_n(S, G)) = B_n(C(S), G)$$

The second part of (10.2) is a trivial consequence of the first part and (10.1). It also contains our main theorem (8.1) as a corollary.

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TORSION AND FLEXURE OF A BEAM WHOSE CROSS-SECTION IS A SECTOR OF A CIRCLE

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1. In this paper a new function-theoretic method has been used to solve the torsion-flexure problem of a beam whose cross-section is a sector of a circle. The method is an extension of the one developed by Ghosh (1947, 1948) and applied by him to the solution of the torsion-flexure problem of beams bounded partly by a straight line. Using the conformal representation $z = \zeta^m$, the sector of a circle is transformed into a semi-circle and the problem is reduced to the determination of a function, analytic within the semi-circle, whose imaginary part vanishes on the bounding diameter and takes up given values on the circumference of the semi-circle. This function is continued analytically by the principle of reflection, to the lower half of the circle of which the semi-circle forms a part. The function is then determined with the help of Schwarz's formula. The results of this paper are in agreement with those given by Stevenson (1938).

TORSION PROBLEM

2. Let the sector be bounded by $r = 1$, $\theta = 0$ and $\theta = 2\delta$, 2δ being the angle of the sector. Then,

$$z = \omega(\zeta) = \zeta^m, \quad m = 2\delta/\pi \quad (2.1)$$

is the transformation formula which represents conformally the given sector on the upper half of the unit circle, with its bounding diameter on the real axis. Let α , β denote the upper and lower semi-circumferences of the unit circle and γ , its circumference. The imaginary part of the complex torsion function $F_0(\zeta)$ has the values $\frac{1}{2}(\zeta\bar{\zeta})^m$ on the boundary. If we choose the function

$$G_0(\zeta) = F_0(\zeta) - \frac{1}{2}(i + \tan 2\delta)\zeta^{2m} \quad (2.2)$$

($2\delta \neq \pi/2, 3\pi/2$), it is singlevalued and analytic within the semicircle and its imaginary part takes the value zero on the real axis and the value

$$\frac{1}{2} - \frac{1}{2}(\zeta^{2m} + \zeta^{-2m}) + \frac{1}{2}i \tan 2\delta(\zeta^{2m} - \zeta^{-2m}), = X(\text{say}) \quad (2.3)$$

on the semi-circular boundary α . By Schwarz's principle of reflection, $G_0(\zeta)$ can be continued analytically to the lower semi-circle so that if $\bar{\zeta}$ be the point on β corresponding to a point ζ on α , the value of the imaginary part of $G_0(\zeta)$ at the point $\bar{\zeta}$ is $-X$. Remembering that on β , $\bar{\zeta} = 1/\zeta$ and applying Schwarz's formula we get,

$$G_c(\xi) = \frac{1}{2\pi} \left[\int_a \frac{dt}{t-\xi} - \int_\beta \right] - \frac{1}{4\pi} \left[\int_a (t^{2m} + t^{-2m}) \frac{dt}{t-\xi} - \int_\beta \right] + \frac{i \tan 2\delta}{4\pi} \int_\gamma (t^{2m} - t^{-2m}) \frac{dt}{t-\xi}. \quad (2.4)$$

Therefore

$$F_0(\xi) = \frac{1}{2}(i + \tan 2\delta)\xi^{2m} + \frac{1}{\pi} \log \frac{1+\xi}{1-\xi} - \frac{1}{4\pi} \left[(1-i \tan 2\delta) \int_a t^{2m} \frac{dt}{t-\xi} \right. \\ \left. + (1+i \tan 2\delta) \int_a t^{-2m} \frac{dt}{t-\xi} - (1-i \tan 2\delta) \int_\beta t^{-2m} \frac{dt}{t-\xi} - (1+i \tan 2\delta) \int_\beta t^{2m} \frac{dt}{t-\xi} \right]. \quad (2.5)$$

This result holds for all admissible values of δ except $\pi/4$ and $3\pi/4$. The paths of the integrals appearing in (2.5) do not pass through any singularity of the integrands. This form of the complex torsion function is more general than that given by Love in integral form which converges only when $\pi/2\delta > 2$. The integrals appearing in (2.5) can be evaluated in the form of infinite series leading to known expressions (Stevenson, 1938) of the torsion function. The torsional rigidity is given by μN_0 where (Ghosh 1947b, p. 108)

$$N_0 = \iint_R \left(x^2 + y^2 - x \frac{\partial \psi}{\partial x} - y \frac{\partial \psi}{\partial y} \right) dx dy, \quad (2.6)$$

the integration being taken over the area of the sector. Dividing N_0 into two parts

$$N_{01} = \iint_R (x^2 + y^2) dx dy = \frac{1}{2}\delta \quad (2.7)$$

and N_{02} , we can transform N_{02} into

$$N_{02} = -\frac{1}{2}R \left[\int_a F_0(\xi) d(\xi\bar{\xi})^m + \int_{-1}^1 F_0(\xi) d|\xi|^{2m} \right]. \quad (2.8)$$

Now we have,

$$\int_{-1}^1 |\xi|^{2m} d|\xi|^{2m} = \frac{1}{2}(1 - e^{4i\delta}) \quad (2.9)$$

$$\int_{-1}^1 \log \frac{1+\xi}{1-\xi} d|\xi|^{2m} = 4 \int_0^1 \left(\xi + \frac{\xi^3}{3} + \frac{\xi^5}{5} + \dots \right) d|\xi|^{2m} \\ = 4 \left[\left(1 - \frac{\pi}{\pi + 4\delta} \right) + \left(\frac{1}{3} - \frac{\pi}{3\pi + 4\delta} \right) + \dots \right] = 2 \left\{ \psi \left(\frac{1}{2} + \frac{2\delta}{\pi} \right) - \psi \left(\frac{1}{2} \right) \right\}. \quad (2.10)$$

Using (Bromwich, 1926, p. 522)

$$\psi(x) - \psi(y) = \sum_0^\infty \left(\frac{1}{n+y} - \frac{1}{n+x} \right) \quad (2.11)$$

where

$$\psi(x) = -\gamma - \sum_0^\infty \left(\frac{1}{n+x} - \frac{1}{n+1} \right), \quad (2.12)$$

γ , being the Euler's constant.

Also remembering that on α , the amplitude of t changes from 0 to π and on β , from $-\pi$ to 0, we get

$$\begin{aligned}
\int_{-1}^1 \int_{\alpha} t^{2m} \frac{dt}{t-\xi} d|\xi|^{2m} &= \int_{\alpha} t^{2m} dt \left[\int_0^1 \frac{2\xi}{t^2-\xi^2} d|\xi|^{2m} \right] \\
&= 2m \int_0^1 t^{2m-2} dt \left[\int_0^1 \left(1 + \frac{\xi^2}{t^2} + \frac{\xi^4}{t^4} + \dots \right) \xi^{2m} d\xi \right] \\
&= 8\pi\delta(1+e^{4i\delta}) \left[\frac{1}{\pi-4\delta} \cdot \frac{1}{\pi+4\delta} + \frac{1}{3\pi-4\delta} \cdot \frac{1}{3\pi+4\delta} + \dots \right] \\
&= \pi(1+e^{4i\delta}) \left[\left(\frac{1}{\pi-4\delta} - \frac{1}{\pi+4\delta} \right) + \left(\frac{1}{3\pi-4\delta} - \frac{1}{3\pi+4\delta} \right) + \dots \right] \\
&= \frac{\pi}{2} (1+e^{4i\delta}) \left\{ \psi\left(\frac{1}{2} + \frac{2\delta}{\pi}\right) - \psi\left(\frac{1}{2} - \frac{2\delta}{\pi}\right) \right\}. \tag{2.13}
\end{aligned}$$

Proceeding as before,

$$\int_{-1}^1 \int_{\beta} t^{2m} \frac{dt}{t-\xi} d|\xi|^{2m} = -\frac{1}{2}(1+e^{-4i\delta}) \left\{ \psi\left(\frac{1}{2} + \frac{2\delta}{\pi}\right) - \psi\left(\frac{1}{2} - \frac{2\delta}{\pi}\right) \right\} \tag{2.14}$$

$$\int_{-1}^1 \int_{\alpha} t^{-2m} \frac{dt}{t-\xi} d|\xi|^{2m} = \frac{2\delta}{\pi} (1+e^{-4i\delta}) \psi'\left(\frac{1}{2} + \frac{2\delta}{\pi}\right) \tag{2.15}$$

$$\int_{-1}^1 \int_{\beta} t^{-2m} \frac{dt}{t-\xi} d|\xi|^{2m} = -\frac{2\delta}{\pi} (1+e^{4i\delta}) \psi'\left(\frac{1}{2} + \frac{2\delta}{\pi}\right) \tag{2.16}$$

where

$$\psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^2} \tag{2.17}$$

is the trigamma function. Hence

$$N_0 = \frac{\delta}{2} + \frac{1}{\pi} \left\{ \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \frac{2\delta}{\pi}\right) + \frac{\delta}{\pi} \psi'\left(\frac{1}{2} + \frac{2\delta}{\pi}\right) \right\}. \tag{2.18}$$

N_0 is continuous when $\delta = \pi/4$ and $3\pi/4$ and can be used for finding the torsional rigidity for these values of δ . Taking δ different from $\pi/4$ and using the expression for N_0 we can make δ tend to $\pi/4$ in this expression. A similar procedure can be applied to the case $\delta = 3\pi/4$.

3. The complex torsion function where $\delta = \pi/4$ can be obtained in the following way. The transformation formula (2.1) now becomes

$$z = \zeta^{\frac{1}{2}} \tag{3.1}$$

Choosing

$$G_0(\zeta) = F_0(\zeta) - \frac{1}{2}i\zeta + (\zeta/\pi) \log \zeta. \tag{3.2}$$

the imaginary part of this function takes the value zero on the real axis in the ζ -plane and the value

$$\frac{1}{2} - \frac{1}{2}(\zeta + 1/\zeta) - (i/2\pi)\{\zeta \log \zeta + (1/\zeta) \log \zeta\} = Y \text{ (say)} \tag{3.3}$$

on the boundary α . Continuing as before the function $G_0(\zeta)$ to the lower semi-circle β and using Schwarz's formula we get

$$F_0(\zeta) - \frac{1}{2}i\zeta + \frac{\zeta}{\pi} \log \zeta = \frac{1}{2\pi} \left[\int_a \frac{dt}{t-\zeta} - \int_\beta \right] - \frac{1}{2\pi} \left[\int_a \frac{t^2+1}{t} \frac{dt}{t-\zeta} - \int_\beta \right] - \frac{i}{2\pi^2} \int_\gamma \left(t \log t + \frac{1}{t} \log t \right) \frac{dt}{t-\zeta}. \quad (8.4)$$

Now

$$\int_a \frac{dt}{t-\zeta} - \int_\beta = 2 \log \frac{1+\zeta}{1-\zeta} \quad (8.5)$$

$$\int_a \frac{t^2+1}{t(t-\zeta)} dt - \int_\beta = 2 \left(\zeta + \frac{1}{\zeta} \right) \log \frac{1+\zeta}{1-\zeta} \quad (8.6)$$

$$\int_\gamma \frac{t^2+1}{t(t-\zeta)} \log t dt = \int_\gamma \left\{ 1 - \frac{1}{\zeta t} + \left(\zeta + \frac{1}{\zeta} \right) \frac{1}{t-\zeta} \right\} \log t dt = 2\pi i \left(\zeta + \frac{1}{\zeta} \right) \log(1+\zeta) \quad (8.7)$$

Therefore

$$F_0(\zeta) = \frac{1}{2}i\zeta + \frac{1}{\pi} \left[-\zeta \log \zeta + \log \frac{1+\zeta}{1-\zeta} + \frac{1}{2} \left(\zeta + \frac{1}{\zeta} \right) \log(1-\zeta^2) \right]. \quad (8.8)$$

This agrees with the result given by Ghosh (1948b, p. 113).

FLEXURE PROBLEM

4. The solution of the flexure problem lies in determining two analytic functions whose imaginary parts ψ_1 and ψ_2 satisfy the following boundary conditions, and in determining the co-ordinates of the centre of flexure. Thus as given by Ghosh (1948a, p. 77)

$$\psi_1 = [(1+\sigma)a^2 - \sigma b^2]y - (1+\sigma)axy + \sigma by^2 - \frac{1}{2}(1+2\sigma)y^3 + (1+\sigma)\mathbf{I} \int_A^P (z-a)\bar{z}dz, \quad (4.1)$$

$$\psi_2 = [\sigma a^2 - (1+\sigma)b^2]x - \sigma ax^2 + (1+\sigma)bxy + \frac{1}{2}\sigma x^3 - (1+\sigma)b\mathbf{I} \int_A^P \bar{z}dz + \frac{1}{2}(1+\sigma)\mathbf{R} \int_A^P (z-\bar{z})^2 dz \quad (4.2)$$

on the boundary, where $a = \sin 2\delta/(\sin \delta)$ and $b = 2 \sin^2 \delta/(\sin \delta)$ denote the co-ordinates of the C.G. of the area of the cross-section of the beam. Let $F_1(\zeta)$ and $F_2(\zeta)$ be complex flexure functions whose imaginary parts satisfy boundary conditions (4.1) and (4.2) and we choose

$$G_1(\zeta) = F_1(\zeta) - \{(1+\sigma)a^2 - \sigma b^2\}\zeta^m + \frac{1}{2}(1+\sigma)a\zeta^{2m} - \frac{\sigma b \sin 4\delta}{2(1+\cos 4\delta)}\zeta^{2m} - \frac{(1+2\sigma)(\sin 6\delta - 3 \sin 2\delta)}{12 \sin 6\delta}\zeta^{3m} - \frac{(1+\sigma) \sin 2\delta}{3 \sin 6\delta}\zeta^{3m} \quad (4.3)$$

$$G_2(\zeta) = F_2(\zeta) - i\{\sigma a^2 - (1+\sigma)b^2\}\zeta^m + \frac{1}{2}i\sigma a(2-i \tan 2\delta)\zeta^{2m} - \frac{1}{2}(1+\sigma)b\zeta^{2m} - \frac{1}{3}\sigma \left\{ \frac{3(\cos 2\delta - \cos 6\delta)}{4 \sin 6\delta} + i \right\} \zeta^{3m} + \frac{(1+\sigma) \sin^2 2\delta \cos 2\delta}{3 \sin 6\delta} \zeta^{3m}. \quad (4.4)$$

The above functions are analytic within the boundary and their imaginary parts are, by (4.1) and (4.2), easily seen to take up the value zero on the real axis of the ζ -plane. The above functions can therefore be analytically continued to the lower semi-circle by the principle of reflection. Also we find from (4.1) and (4.3) that the imaginary part of $G_1(\zeta)$ takes up on the boundary α , the modified boundary value

$$\begin{aligned} & -\frac{1}{2}\sigma b(\zeta^{2m} + \zeta^{-2m} - 2) - \frac{\sigma b \sin 4\delta}{4i(1 + \cos 4\delta)}(\zeta^{2m} - \zeta^{-2m}) + \frac{(1+2\sigma)}{24i}(\zeta^{3m} - \zeta^{-3m} - 3\zeta^m + 3\zeta^{-m}) \\ & + \frac{i(1+2\sigma)(\sin 6\delta - 3 \sin 2\delta)}{24 \sin 6\delta}(\zeta^{3m} - \zeta^{-3m}) + \frac{(1+\sigma)}{2i}\{F(\zeta) - \bar{F}(1/\zeta)\} \\ & - \frac{(1+\sigma) \sin 2\delta}{6i \sin 6\delta}(\zeta^{3m} - \zeta^{-3m}) = X_1 \text{ (say)} \quad (4.5) \end{aligned}$$

where

$$F(\zeta) = \int_1^{\zeta} (\zeta^m - a)\zeta^{-m} d(\zeta^m) = \zeta^m - am \log \zeta - 1 \quad (4.6)$$

and the imaginary part of $G_2(\zeta)$ will by (4.2) and (4.4), take up on the boundary α , the value

$$\begin{aligned} & -\frac{1}{2}\sigma a(\zeta^{2m} + \zeta^{-2m} + 2) - \frac{1}{2}ib(1+\sigma)(\zeta^{2m} - \zeta^{-2m}) + (\sigma/24)(\zeta^{3m} + \zeta^{-3m} + 3\zeta^m + 3\zeta^{-m}) \\ & + \frac{1}{2}i(1+\sigma)b\{G(\zeta) - \bar{G}(1/\zeta)\} + \frac{1}{8}(1+\sigma)\{H(\zeta) + \bar{H}(1/\zeta)\} + \frac{1}{2}\sigma a\{\zeta^{2m} + \zeta^{-2m} \\ & + \frac{1}{1 + \cos 4\delta}(\zeta^{2m}e^{-4i\delta} + \zeta^{-2m}e^{4i\delta} + \zeta^{2m} + \zeta^{-2m})\} - \frac{1}{8}\sigma\left\{\frac{3(\cos 2\delta - \cos 6\delta)}{8i \sin 6\delta}(\zeta^{3m} - \zeta^{-3m})\right. \\ & \left. + \frac{1}{2}(\zeta^{3m} + \zeta^{-3m})\right\} + \frac{(1+\sigma) \sin^2 2\delta \cos 2\delta}{6i \sin 6\delta}(\zeta^{3m} - \zeta^{-3m}) = X_2 \text{ (say)} \quad (4.7) \end{aligned}$$

where

$$G(\zeta) = \int_1^{\zeta} \bar{z} dz = m \log \zeta, \quad H(\zeta) = \int_1^{\zeta} (z - \bar{z})^2 dz = \frac{1}{3}\zeta^{3m} - \zeta^{-m} - 2\zeta^m + \frac{8}{3}. \quad (4.8)$$

Therefore for the point $\bar{\zeta} = 1/\zeta$, on β the imaginary parts of G_1 and G_2 will be respectively equal to $-X_1$ and $-X_2$. Hence applying Schwarz's formula and using (3.5) we get

$$\begin{aligned} G_1(\zeta) &= \frac{\sigma b}{\pi} \log \frac{1+\zeta}{1-\zeta} - 2ma(1+\sigma) \log(1+\zeta) - \frac{\sigma b}{4\pi} \left[\int_{\alpha} (t^{2m} + t^{-2m}) \frac{dt}{t-\zeta} - \int_{\beta} \right] \\ & - \frac{\sigma b \sin 4\delta}{4\pi i(1 + \cos 4\delta)} \int_{\gamma} (t^{2m} - t^{-2m}) \frac{dt}{t-\zeta} + \frac{(2\sigma-1) \sin 2\delta}{24\pi i \sin 6\delta} \int_{\gamma} (t^{3m} - t^{-3m}) \frac{dt}{t-\zeta} \\ & - \frac{i(2\sigma+3)}{8\pi} \int_{\gamma} (t^m - t^{-m}) \frac{dt}{t-\zeta}, \quad (4.9) \end{aligned}$$

$$G_2(\zeta) = \left\{ \frac{4(1+\sigma)}{8\pi} - \frac{\sigma a}{\pi} \right\} \log \frac{1+\zeta}{1-\zeta} - 2mb(1+\sigma) \log(1+\zeta)$$

$$- \frac{3+2\sigma}{8\pi} \left[\int_{\alpha} (t^m + t^{-m}) \frac{dt}{t-\zeta} - \int_{\beta} \right] + \frac{\sigma a}{4\pi} \left[\left\{ \int_{\alpha} (t^{2m} + t^{-2m}) \frac{dt}{t-\zeta} - \int_{\beta} \right\} \right]$$

$$-i \tan 2\delta \int_{\gamma} (t^{3m} - t^{-3m}) \frac{dt}{t-\zeta} \Big] + \frac{1}{8\pi} \left\{ \frac{1}{3} (1+\sigma) - \sigma a \right\} \left[\int_a (t^{3m} + t^{-3m}) \frac{dt}{t-\zeta} - \int_{\beta} \right] \\ + \frac{(1-2\sigma) \sin^2 2\delta \cos 2\delta}{6\pi i \sin 6\delta} \int_{\gamma} (t^{3m} - t^{-3m}) \frac{dt}{t-\zeta}. \quad (4.10)$$

The integrals appearing in $G_1(\zeta)$ and $G_2(\zeta)$ can, if necessary, be evaluated in infinite series. The expression for $F_1(\zeta)$, $F_2(\zeta)$ are valid for all values of δ except $\pi/6$, $\pi/4$, $\pi/3$, $3\pi/4$.

8. We now find the flexural moments in the form given by Ghosh (1948a, p. 79).

We write

$$N_1 = N_{11} + N_{12} + N_{13}, \quad N_2 = N_{21} + N_{22} + N_{23} \quad (5.1)$$

where,

$$N_{11} = \iint_R \{ (1+\sigma)(x-a)^2(y-b) - \sigma(y-b)^3 \} dx dy \quad (5.2)$$

$$N_{21} = \iint_R \{ \sigma(x-a)^3 - (1+\sigma)(x-a)(y-b)^2 \} dx dy. \quad (5.3)$$

$$N_{r2} = - \iint_R \left(x \frac{\partial \psi_r}{\partial x} + y \frac{\partial \psi_r}{\partial y} \right) dx dy \\ = - \frac{1}{2} \mathbf{R} \int_a F_r(\zeta) d[\omega(\zeta) \bar{\omega}(1/\zeta)] - \frac{1}{2} \int_{-1}^1 F_r(\xi) d[\omega(\xi) \bar{\omega}(\xi)]. \quad (5.4)$$

$$N_{r3} = \iint_R \left(a \frac{\partial \psi_r}{\partial x} + b \frac{\partial \psi_r}{\partial y} \right) dx dy = \frac{1}{2} \mathbf{R}(a+ib) \left[\int_a F_r(\zeta) d\{\omega(1/\zeta)\} + \int_{-1}^1 F_r(\xi) d\bar{\omega}(\xi) \right]. \quad (5.5)$$

We get at once

$$N_{11} = (1+\sigma) \left[\frac{1}{16} (1 - \cos^3 2\delta) - b \left(\frac{\delta}{4} + \frac{\sin 4\delta}{16} \right) - \frac{a}{4} \sin^2 2\delta + 2a^2 b \delta \right] \\ - \sigma \left[\frac{1}{16} (3 \sin^2 \delta - \frac{1}{3} \sin^2 3\delta) - 3b \left(\frac{\delta}{4} - \frac{\sin 4\delta}{16} \right) + 2b^3 \delta \right], \quad (5.6)$$

$$N_{21} = (1+\sigma) \left[-\frac{\sin^3 2\delta}{15} + \frac{1}{4} b \sin^3 2\delta - 2ab^2 \delta + a \left(\frac{1}{4} \delta - \frac{\sin 4\delta}{16} \right) \right] \\ + \sigma \left[\frac{\sin 6\delta + 9 \sin 2\delta}{60} - 3a \left(\frac{1}{4} \delta + \frac{\sin 4\delta}{16} \right) + 2a^3 \delta \right]. \quad (5.7)$$

Substituting for $F_1(\zeta)$, $F_2(\zeta)$ we find

$$N_{12} = - \iint_R x \left[-\{(1+\sigma)a - \sigma b \tan 2\delta\}y + \left\{ \frac{(1+\sigma) \sin 4\delta \cos 2\delta}{\sin 6\delta} - \frac{2\sigma \sin^3 2\delta}{\sin 6\delta} \right\} xy \right] dx dy \\ - \iint_R y \left[\{(1+\sigma)a^2 - \sigma b^2\} - \{(1+\sigma)a - \sigma b \tan 2\delta\}x + \left\{ \frac{(1+\sigma) \sin 4\delta \cos 2\delta}{2 \sin 6\delta} \right. \right. \\ \left. \left. - \frac{\sigma \sin^3 2\delta}{\sin 6\delta} \right\} (x^2 - y^2) \right] dx dy - \frac{1}{2} \mathbf{R} \int_{-1}^1 G_1(\xi) d|\xi|^{2m}. \quad (5.8)$$

$$\begin{aligned}
N_{22} = & - \int \int_R x [\{\sigma a^2 - (1 + \sigma)b^2\} - \sigma a(2x + y \tan 2\delta) \\
& + (1 + \sigma)by + \sigma \left\{ \frac{3 \sin 4\delta \sin 2\delta}{\sin 6\delta} xy + x^2 - y^2 \right\} - \frac{2(1 + \sigma) \sin^2 2\delta \cos 2\delta}{\sin 6\delta} xy] dx dy \\
& - \int \int_R y [-\sigma a(-2y + x \tan 2\delta) + b(1 + \sigma)x + \sigma \left\{ \frac{3 \sin 4\delta \sin 2\delta}{2 \sin 6\delta} (x^2 - y^2) - 2xy \right\} \\
& - \frac{(1 + \sigma) \sin^2 2\delta \cos 2\delta}{\sin 6\delta} (x^2 - y^2)] dx dy - \frac{1}{2} \mathbf{R} \int_{-1}^1 G_2(\xi) d|\xi|^m. \quad (5.9)
\end{aligned}$$

$$\begin{aligned}
N_{13} = & \int \int_R a [-\{(1 + \sigma)a - \sigma b \tan 2\delta\}y \\
& + \left\{ \frac{(1 + \sigma) \sin 4\delta \cos 2\delta}{\sin 6\delta} - \frac{2\sigma \sin^3 2\delta}{\sin 6\delta} \right\} xy] dx dy + \int \int_R b [\{(1 + \sigma)a^2 - \sigma b^2\} \\
& - \{(1 + \sigma)a - \sigma b \tan 2\delta\}x + \left\{ \frac{(1 + \sigma) \sin 4\delta \cos 2\delta}{2 \sin 6\delta} - \frac{\sigma \sin^3 2\delta}{\sin 6\delta} \right\} (x^2 - y^2)] dx dy \\
& + \frac{1}{2} \mathbf{R}(a + ib) \left[\int_a G_1(\xi) d(\xi^{-m}) + \int_{-1}^1 G_1(\xi) d\{\overline{\omega(\xi)}\} \right]. \quad (5.10)
\end{aligned}$$

$$\begin{aligned}
N_{23} = & \int \int_R a [\{\sigma a^2 - (1 + \sigma)b^2\} - \sigma a(2x + y \tan 2\delta) + (1 + \sigma)by \\
& + \sigma \left\{ \frac{3 \sin 4\delta \sin 2\delta}{\sin 6\delta} xy + x^2 - y^2 \right\} - \frac{2(1 + \sigma) \sin^2 2\delta \cos 2\delta}{\sin 6\delta} xy] dx dy \\
& + \int \int_R b [-\sigma a(-2y + x \tan 2\delta) + (1 + \sigma)bx + \sigma \left\{ \frac{3 \sin 4\delta \sin 2\delta}{2 \sin 6\delta} (x^2 - y^2) - 2xy \right\} \\
& - (1 + \sigma) \frac{\sin^2 2\delta \cos 2\delta}{\sin 6\delta} (x^2 - y^2)] dx dy + \frac{1}{2} \mathbf{R}(a + ib) \left[\int_a G_2(\xi) d(\xi^{-m}) \right. \\
& \left. + \int_{-1}^1 G_2(\xi) d\{\overline{\omega(\xi)}\} \right]. \quad (5.11)
\end{aligned}$$

The first two integrals of each of (5.8), (5.9), (5.10), (5.11) can be evaluated very easily and we proceed to evaluate the line integrals in the two following sections.

6. Let us denote

$$\begin{aligned}
p_r(\xi) &= \int t^{rm} \frac{dt}{t - \xi}, & q_r(\xi) &= \int t^{-rm} \frac{dt}{t - \xi}, \\
u_r(\xi) &= \int t^{rm} \frac{dt}{t - \xi}, & v_r(\xi) &= \int t^{-rm} \frac{dt}{t - \xi}, \quad r = 1, 2, 3. \quad (6.1)
\end{aligned}$$

Proceeding exactly as in §2, we get,

$$\int_{-1}^1 p_1(\xi) d|\xi|^{2m} = \frac{2}{3}(1 + e^{2i\delta}) \{ \psi(\frac{1}{2} + m) - \psi(\frac{1}{2} - \delta/\pi) \} \quad (6.2)$$

$$\int_{-1}^1 q_1(\xi) d|\xi|^{2m} = 2(1 + e^{-2i\delta}) \{ \psi(\frac{1}{2} + m) - \psi(\frac{1}{2} + \delta/\pi) \} \quad (6.3)$$

$$\int_{-1}^1 u_1(\xi) d|\xi|^{2m} = -\frac{2}{3}(1 + e^{-2i\delta}) \{ \psi(\frac{1}{2} + m) - \psi(\frac{1}{2} - \delta/\pi) \} \quad (6.4)$$

$$\int_{-1}^1 v_1(\xi) d|\xi|^{2m} = -2(1 + e^{2i\delta}) \{ \psi(\frac{1}{2} + m) - \psi(\frac{1}{2} + \delta/\pi) \} \quad (6.5)$$

$$\int_{-1}^1 p_2(\xi) d|\xi|^{2m} = \frac{1}{2}(1 + e^{4i\delta}) \{ \psi(\frac{1}{2} + m) - \psi(\frac{1}{2} - m) \} = \frac{1}{2}\pi(1 + e^{4i\delta}) \tan 2\delta \quad (6.6)$$

$$\int_{-1}^1 q_2(\xi) d|\xi|^{2m} = m(1 + e^{-4i\delta}) \psi'(\frac{1}{2} + m) \quad (6.7)$$

$$\int_{-1}^1 u_2(\xi) d|\xi|^{2m} = -\frac{1}{2}\pi(1 + e^{-4i\delta}) \tan 2\delta \quad (6.8)$$

$$\int_{-1}^1 v_2(\xi) d|\xi|^{2m} = -m(1 + e^{4i\delta}) \psi'(\frac{1}{2} + m) \quad (6.9)$$

$$\int_{-1}^1 p_3(\xi) d|\xi|^{2m} = \frac{2}{3}(1 + e^{6i\delta}) \{ \psi(\frac{1}{2} + m) - \psi(\frac{1}{2} - 3\delta/\pi) \} \quad (6.10)$$

$$\int_{-1}^1 q_3(\xi) d|\xi|^{2m} = 2(1 + e^{-6i\delta}) \{ \psi(\frac{1}{2} + 3\delta/\pi) - \psi(\frac{1}{2} + m) \} \quad (6.11)$$

$$\int_{-1}^1 u_3(\xi) d|\xi|^{2m} = -\frac{2}{3}(1 + e^{-6i\delta}) \{ \psi(\frac{1}{2} + m) - \psi(\frac{1}{2} - 3\delta/\pi) \} \quad (6.12)$$

$$\int_{-1}^1 v_3(\xi) d|\xi|^{2m} = -2(1 + e^{6i\delta}) \{ \psi(\frac{1}{2} + 3\delta/\pi) - \psi(\frac{1}{2} + m) \} \quad (6.13)$$

whence,

$$\begin{aligned} \int_{-1}^1 \left[\int_{\gamma} (t^m - t^{-m}) \frac{dt}{t-\xi} \right] d|\xi|^{2m} &= \int_{-1}^1 \{ (p_1 + u_1) - (q_1 + v_1) \} d|\xi|^{2m} \\ &= 4i \sin 2\delta \left[\frac{4}{3} \psi(\frac{1}{2} + m) - \frac{1}{3} \psi(\frac{1}{2} - \delta/\pi) - \psi(\frac{1}{2} + \delta/\pi) \right]. \end{aligned} \quad (6.14)$$

$$\begin{aligned} \int_{-1}^1 \left[\int_{\alpha} (t^m + t^{-m}) \frac{dt}{t-\xi} - \int_{\beta} \right] d|\xi|^{2m} &= \int_{-1}^1 \{ (p_1 + q_1) - (u_1 + v_1) \} d|\xi|^{2m} \\ &= 4(1 + \cos 2\delta) \left[\frac{4}{3} \psi(\frac{1}{2} + m) - \frac{1}{3} \psi(\frac{1}{2} - \delta/\pi) - \psi(\frac{1}{2} + \delta/\pi) \right] \end{aligned} \quad (6.15)$$

$$\begin{aligned} \int_{-1}^1 \left[\int_{\gamma} (t^{2m} - t^{-2m}) \frac{dt}{t-\xi} \right] d|\xi|^{2m} &= \int_{-1}^1 \{ (p_2 + u_2) - (q_2 + v_2) \} d|\xi|^{2m} \\ &= 2i \sin 4\delta \left[m \psi'(\frac{1}{2} + m) + \frac{1}{2}\pi \tan 2\delta \right] \end{aligned} \quad (6.16)$$

$$\int_{-1}^1 \left[\int_{\alpha} (t^{2m} + t^{-2m}) \frac{dt}{t-\xi} - \int_{\beta} d|\xi|^{2m} \right] = \int_{-1}^1 \{(p_2 + q_2) - (u_2 + v_2)\} d|\xi|^{2m} \\ = 2(1 + \cos 4\delta) \left[m\psi'(\tfrac{1}{2} + m) + \tfrac{1}{2}\pi \tan 2\delta \right] \quad (6.17)$$

$$\int_{-1}^1 \left[\int_{\gamma} (t^{2m} - t^{-2m}) \frac{dt}{t-\xi} \right] d|\xi|^{2m} = \int_{-1}^1 \{(p_3 + u_3) - (q_3 + v_3)\} d|\xi|^{2m} \\ = 2i \sin 6\delta \left[2\psi(\tfrac{1}{2} + 3\delta/\pi) - \tfrac{8}{3}\psi(\tfrac{1}{2} + m) - \tfrac{8}{3}\psi(\tfrac{1}{2} - 3\delta/\pi) \right] \quad (6.18)$$

$$\int_{-1}^1 \left[\int_{\alpha} (t^{2m} + t^{-2m}) \frac{dt}{t-\xi} - \int_{\beta} d|\xi|^{2m} \right] = \int_{-1}^1 \{(p_3 + q_3) - (u_3 + v_3)\} d|\xi|^{2m} \\ = 2(1 + \cos 6\delta) \left[2\psi(\tfrac{1}{2} + 3\delta/\pi) - \tfrac{8}{3}\psi(\tfrac{1}{2} + m) - \tfrac{8}{3}\psi(\tfrac{1}{2} - 3\delta/\pi) \right]. \quad (6.19)$$

Also

$$\int_{-1}^1 \log(1 + \xi) d|\xi|^{2m} = \psi(\tfrac{1}{2} + m) - \psi(\tfrac{1}{2}). \quad (6.20)$$

Using these results, we get

$$N_{12} = (1 + \sigma) \left[\tfrac{1}{2}a \sin^2 2\delta - \frac{2a^2 \sin^2 \delta}{3} - \frac{\sin^2 3\delta}{30} - \frac{\sin^2 \delta}{2} \right. \\ \left. + \frac{\sin 2\delta}{3\pi} \{3\psi(\tfrac{1}{2} + \delta/\pi) + \tfrac{1}{3}\psi(\tfrac{1}{2} + 3\delta/\pi) - \tfrac{8}{3}\psi(\tfrac{1}{2} + m) - 2\psi(\tfrac{1}{2})\} \right] + \sigma \left[\tfrac{1}{8}b \sin 4\delta \right. \\ \left. + \frac{2b^2 \sin^2 \delta}{3} - \frac{\sin^2 3\delta}{30} + \frac{\sin^2 \delta}{6} + \frac{b}{2\pi} \{m\psi'(\tfrac{1}{2} + m) - 2\psi(\tfrac{1}{2} + m) + 2\psi(\tfrac{1}{2})\} \right. \\ \left. + \frac{\sin 2\delta}{\pi} \{-\tfrac{1}{3}\psi(\tfrac{1}{2} + 3\delta/\pi) - \tfrac{1}{3}\psi(\tfrac{1}{2} + \delta/\pi) + \tfrac{8}{15}\psi(\tfrac{1}{2} + m)\} \right]. \quad (6.21)$$

$$N_{22} = (1 + \sigma) \left[\tfrac{1}{3}b^2 \sin 2\delta - \tfrac{1}{2}b \sin^2 2\delta - \frac{\sin 6\delta}{60} + \tfrac{1}{2} \sin 2\delta \right. \\ \left. - \frac{\cos^2 \delta}{6\pi} \{\tfrac{4}{3}\psi(\tfrac{1}{2} + 3\delta/\pi) - \tfrac{4}{3}\psi(\tfrac{1}{2} + m)\} + \frac{2 \cos^2 \delta}{3\pi} \{\psi(\tfrac{1}{2} + m) - 3\psi(\tfrac{1}{2} + \delta/\pi) + 2\psi(\tfrac{1}{2})\} \right] \\ + \sigma \left[-\tfrac{1}{3}a^2 \sin 2\delta + \tfrac{1}{3}a \sin 4\delta - \frac{\sin 6\delta}{60} - \frac{\sin 2\delta}{12} - \frac{a\delta}{\pi^2} \psi'(\tfrac{1}{2} + m) + (a/\pi) \{\psi(\tfrac{1}{2} + m) - \psi(\tfrac{1}{2})\} \right. \\ \left. + \frac{2 \cos^2 \delta}{15\pi} \{3\psi(\tfrac{1}{2} + 3\delta/\pi) - 8\psi(\tfrac{1}{2} + m) + 5\psi(\tfrac{1}{2} + \delta/\pi)\} \right]. \quad (6.22)$$

7. We proceed to evaluate the last integral in each of (5.10) and (5.11) and thence find out N_{13} and N_{23} . We find the following:

$$\int_{\alpha} p_1(\xi) d\omega(1/\xi) = 2\pi\delta \left[\frac{\sin^2 \delta}{\delta^2} + \frac{\cos^2 \delta}{\pi^2} \psi'(\tfrac{1}{2} - \delta/\pi) + \frac{\sin^2 \delta}{\pi^2} \psi'(1 - \delta/\pi) \right] \quad (7.1)$$

$$\int_{\alpha} q_1(\xi) d\omega(1/\xi) = \pi \sin \delta \left[\frac{\sin \delta}{\delta} + 2 \cos \delta \right] e^{-2i\delta} = - \int_{-1}^1 u_1(\xi) d\overline{\omega(\xi)} \quad (7.2)$$

$$\int_a u_1(\zeta) d\omega(1/\zeta) = -2\pi\delta \left[-\frac{\sin^2 \delta}{\delta^2} + \frac{\cos^2 \delta}{\pi^2} \psi'(\tfrac{1}{2} - \delta/\pi) - \frac{\sin^2 \delta}{\pi^2} \psi'(1 - \delta/\pi) \right] e^{-2i\delta} \quad (7.3)$$

$$\int_a v_1(\zeta) d\omega(1/\zeta) = \pi \frac{\sin^2 \delta}{\delta} = - \int_{-1}^1 p_1(\xi) d\overline{\omega}(\xi) \quad (7.4)$$

$$\int_{-1}^1 q_1(\xi) d\overline{\omega}(\xi) = 2\pi\delta \left[-\frac{\sin^2 \delta}{\delta^2} + \frac{\cos^2 \delta}{\pi^2} \psi'(\tfrac{1}{2} + \delta/\pi) - \frac{\sin^2 \delta}{\pi^2} \psi'(1 + \delta/\pi) \right] e^{-2i\delta} \quad (7.5)$$

$$\int_{-1}^1 v_1(\xi) d\overline{\omega}(\xi) = -2\pi\delta \left[\frac{\sin^2 \delta}{\delta^2} + \frac{\cos^2 \delta}{\pi^2} \psi'(\tfrac{1}{2} + \delta/\pi) + \frac{\sin^2 \delta}{\pi^2} \psi'(1 + \delta/\pi) \right] \quad (7.6)$$

$$\begin{aligned} \int_a p_1(\zeta) d\omega(1/\zeta) = \pi e^{i\delta} \left[\frac{\sin \delta \sin 2\delta}{\delta} + \frac{2 \cos \delta \cos 2\delta}{\pi} \{ \psi(\tfrac{1}{2} - \delta/\pi) - \psi(\tfrac{1}{2} - m) \} \right. \\ \left. + \frac{2 \sin \delta \sin 2\delta}{\pi} \{ \psi(1 - \delta/\pi) - \psi(1 - m) \} \right] \quad (7.7) \end{aligned}$$

$$\begin{aligned} \int_a q_1(\zeta) d\omega(1/\zeta) = \tfrac{1}{3} \pi e^{-i\delta} \left[\frac{3 \sin \delta \sin 2\delta}{\delta} - \frac{2 \cos \delta \cos 2\delta}{\pi} \{ \psi(\tfrac{1}{2} - \delta/\pi) - \psi(\tfrac{1}{2} + m) \} \right. \\ \left. - \frac{2 \sin \delta \sin 2\delta}{\pi} \{ \psi(1 + m) - \psi(1 - \delta/\pi) \} \right] \quad (7.8) \end{aligned}$$

$$\begin{aligned} \int_a u_2(\zeta) d\omega(1/\zeta) = \pi e^{-i\delta} \left[\frac{\sin \delta \sin 2\delta}{\delta} - \frac{2 \cos \delta \cos 2\delta}{\pi} \{ \psi(\tfrac{1}{2} - \delta/\pi) - \psi(\tfrac{1}{2} - m) \} \right. \\ \left. + \frac{2 \sin \delta \sin 2\delta}{\pi} \{ \psi(1 - \delta/\pi) - \psi(1 - m) \} \right] \quad (7.9) \end{aligned}$$

$$\begin{aligned} \int_a v_2(\zeta) d\omega(1/\zeta) = \tfrac{1}{3} \pi e^{i\delta} \left[\frac{3 \sin \delta \sin 2\delta}{\delta} - \frac{2 \cos \delta \cos 2\delta}{\pi} \{ \psi(\tfrac{1}{2} + m) - \psi(\tfrac{1}{2} - \delta/\pi) \} \right. \\ \left. - \frac{2 \sin \delta \sin 2\delta}{\pi} \{ \psi(1 + m) - \psi(1 - \delta/\pi) \} \right] \quad (7.10) \end{aligned}$$

$$\begin{aligned} \int_{-1}^1 p_2(\xi) d\overline{\omega}(\xi) = \tfrac{1}{3} \pi e^{i\delta} \left[\frac{-3 \sin \delta \sin 2\delta}{\delta} + \frac{2 \cos \delta \cos 2\delta}{\pi} \{ \psi(\tfrac{1}{2} + \delta/\pi) - \psi(\tfrac{1}{2} - m) \} \right. \\ \left. + \frac{2 \sin \delta \sin 2\delta}{\pi} \{ \psi(1 + \delta/\pi) - \psi(1 - m) \} \right] \quad (7.11) \end{aligned}$$

$$\begin{aligned} \int_{-1}^1 q_2(\xi) d\overline{\omega}(\xi) = \pi e^{-i\delta} \left[\frac{-\sin \delta \sin 2\delta}{\delta} + \frac{2 \cos \delta \cos 2\delta}{\pi} \{ \psi(\tfrac{1}{2} + m) - \psi(\tfrac{1}{2} + \delta/\pi) \} \right. \\ \left. - \frac{2 \sin \delta \sin 2\delta}{\pi} \{ \psi(1 + m) - \psi(1 + \delta/\pi) \} \right] \quad (7.12) \end{aligned}$$

$$\int_{-1}^1 u_2(\xi) d\overline{\omega(\xi)} = \frac{1}{8}\pi e^{-2i\delta} \left[\frac{-8 \sin \delta \sin 2\delta}{\delta} - \frac{2 \cos \delta \cos 2\delta}{\pi} \{ \psi(\frac{1}{2} + \delta/\pi) - \psi(\frac{1}{2} - m) \} \right. \\ \left. + \frac{2 \sin \delta \sin 2\delta}{\pi} \{ \psi(1 + \delta/\pi) - \psi(1 - m) \} \right] \quad (7.13)$$

$$\int_{-1}^1 v_2(\xi) d\overline{\omega(\xi)} = -\pi e^{i\delta} \left[\frac{\sin \delta \sin 2\delta}{\delta} + \frac{2 \cos \delta \cos 2\delta}{\pi} \{ \psi(\frac{1}{2} + m) - \psi(\frac{1}{2} + \delta/\pi) \} \right. \\ \left. + \frac{2 \sin \delta \sin 2\delta}{\pi} \{ \psi(1 + m) - \psi(1 + \delta/\pi) \} \right] \quad (7.14)$$

$$\int_{\alpha} p_3(\zeta) d\omega(1/\zeta) = \frac{1}{2}\pi e^{2i\delta} \left[\frac{4 \sin \delta \sin 3\delta}{3\delta} + \frac{2 \cos \delta \cos 3\delta}{\pi} \{ \psi(\frac{1}{2} - \delta/\pi) - \psi(\frac{1}{2} - 3\delta/\pi) \} \right. \\ \left. + \frac{2 \sin \delta \sin 3\delta}{\pi} \{ \psi(1 - \delta/\pi) - \psi(1 - 3\delta/\pi) \} \right] \quad (7.15)$$

$$\int_{\alpha} q_3(\zeta) d\omega(1/\zeta) = \frac{1}{2}\pi e^{-4i\delta} \left[\frac{8 \sin \delta \sin 3\delta}{3\delta} + \frac{2 \cos \delta \cos 3\delta}{\pi} \{ \psi(\frac{1}{2} + 3\delta/\pi) - \psi(\frac{1}{2} - \delta/\pi) \} \right. \\ \left. - \frac{2 \sin \delta \sin 3\delta}{\pi} \{ \psi(1 + 3\delta/\pi) - \psi(1 - \delta/\pi) \} \right] \quad (7.16)$$

$$\int_{\alpha} u_3(\zeta) d\omega(1/\zeta) = \frac{1}{2}\pi e^{-4i\delta} \left[\frac{4 \sin \delta \sin 3\delta}{3\delta} - \frac{2 \cos \delta \cos 3\delta}{\pi} \{ \psi(\frac{1}{2} - \delta/\pi) - \psi(\frac{1}{2} - 3\delta/\pi) \} \right. \\ \left. + \frac{2 \sin \delta \sin 3\delta}{\pi} \{ \psi(1 - \delta/\pi) - \psi(1 - 3\delta/\pi) \} \right] \quad (7.17)$$

$$\int_{\alpha} v_3(\zeta) d\omega(1/\zeta) = \frac{1}{2}\pi e^{2i\delta} \left[\frac{8 \sin \delta \sin 3\delta}{3\delta} - \frac{2 \cos \delta \cos 3\delta}{\pi} \{ (7 + 3\delta/\pi) - \psi(\frac{1}{2} - \delta/\pi) \} \right. \\ \left. - \frac{2 \sin \delta \sin 3\delta}{\pi} \{ \psi(1 + 3\delta/\pi) - \psi(1 - \delta/\pi) \} \right] \quad (7.18)$$

$$\int_1^1 p_3(\xi) d\overline{\omega(\xi)} = \frac{1}{2}\pi e^{2i\delta} \left[\frac{-8 \sin \delta \sin 3\delta}{3\delta} + \frac{2 \cos \delta \cos 3\delta}{\pi} \{ \psi(\frac{1}{2} + \delta/\pi) - \psi(\frac{1}{2} - 3\delta/\pi) \} \right. \\ \left. + \frac{2 \sin \delta \sin 3\delta}{\pi} \{ \psi(1 + \delta/\pi) - \psi(1 - 3\delta/\pi) \} \right] \quad (7.19)$$

$$\int_1^1 q_3(\xi) d\overline{\omega(\xi)} = \frac{1}{2}\pi e^{-4i\delta} \left[-\frac{4 \sin \delta \sin 3\delta}{3\delta} + \frac{2 \cos \delta \cos 3\delta}{\pi} \{ \psi(\frac{1}{2} + 3\delta/\pi) - \psi(\frac{1}{2} + \delta/\pi) \} \right. \\ \left. - \frac{2 \sin \delta \sin 3\delta}{\pi} \{ \psi(1 + 3\delta/\pi) - \psi(1 + \delta/\pi) \} \right] \quad (7.20)$$

$$\int_{-1}^1 u_3(\xi) \overline{d\omega(\xi)} = \frac{1}{2} \pi e^{-4i\delta} \left[-\frac{8 \sin \delta \sin 3\delta}{3\delta} - \frac{2 \cos \delta \cos 3\delta}{\pi} \{ \psi(\frac{1}{2} + \delta/\pi) - \psi(\frac{1}{2} - 3\delta/\pi) \} \right. \\ \left. + \frac{2 \sin \delta \sin 3\delta}{\pi} \{ \psi(1 + \delta/\pi) - \psi(1 - 3\delta/\pi) \} \right] \quad (7.21)$$

$$\int_{-1}^1 v_3(\xi) \overline{d\omega(\xi)} = -\frac{1}{2} \pi e^{2i\delta} \left[\frac{4 \sin \delta \sin 3\delta}{3\delta} + \frac{2 \cos \delta \cos 3\delta}{\pi} \{ \psi(\frac{1}{2} + 3\delta/\pi) - \psi(\frac{1}{2} + \delta/\pi) \} \right. \\ \left. + \frac{2 \sin \delta \sin 3\delta}{\pi} \{ \psi(1 + 3\delta/\pi) - \psi(1 + \delta/\pi) \} \right] \quad (7.22)$$

$$\mathbf{R} \frac{a+ib}{i} \left[\int_{\alpha} \left\{ \int_{\gamma} (t^m - t^{-m}) \frac{dt}{t-\zeta} \right\} d\omega(1/\zeta) + \int_{-1}^1 \{ \quad \} \overline{d\omega(\xi)} \right] \\ = 4\pi r \sin \delta \cos \delta [\sin \delta + \delta \sec \delta] \quad (7.23)$$

$$\mathbf{R} (a+ib) \left[\int_{\alpha} \left\{ \int_{\alpha} (t^m + t^{-m}) \frac{dt}{t-\zeta} - \int_{\beta} \right\} d\omega(1/\zeta) + \int_{-1}^1 \{ \quad \} \overline{d\omega(\xi)} \right] \\ = 4\pi r \cos^2 \delta [\sin \delta + \delta \sec \delta] \quad (7.24)$$

$$\mathbf{R} \frac{a+ib}{i} \left[\int_{\alpha} \left\{ \int_{\gamma} (t^{2m} - t^{-2m}) \frac{dt}{t-\zeta} \right\} d\omega(1/\zeta) + \int_{-1}^1 \{ \quad \} \overline{d\omega(\xi)} \right] \\ = 4\pi r \cos \delta \sin 2\delta \cos 2\delta \left[\frac{4}{3} \tan 2\delta - \frac{2}{3} \tan \delta \right] \quad (7.25)$$

$$\mathbf{R} (a+ib) \left[\int_{\alpha} \left\{ \int_{\alpha} (t^{2m} + t^{-2m}) \frac{dt}{t-\zeta} - \int_{\beta} \right\} d\omega(1/\zeta) + \int_{-1}^1 \{ \quad \} \overline{d\omega(\xi)} \right] \\ = 4\pi r \cos \delta \cos^2 2\delta \left[\frac{4}{3} \tan 2\delta - \frac{2}{3} \tan \delta \right] \quad (7.26)$$

$$\mathbf{R} \frac{a+ib}{i} \left[\int_{\alpha} \left\{ \int_{\gamma} (t^{3m} - t^{-3m}) \frac{dt}{t-\zeta} \right\} d\omega(1/\zeta) + \int_{-1}^1 \{ \quad \} \overline{d\omega(\xi)} \right] \\ = \pi r \cos \delta \sin 3\delta \cos 3\delta [3 \tan 3\delta - \tan \delta] \quad (7.27)$$

$$\mathbf{R} (a+ib) \left[\int_{\alpha} \left\{ \int_{\alpha} (t^{3m} + t^{-3m}) \frac{dt}{t-\zeta} - \int_{\beta} \right\} d\omega(1/\zeta) + \int_{-1}^1 \{ \quad \} \overline{d\omega(\xi)} \right] \\ = \pi r \cos \delta \cos^2 3\delta [3 \tan 3\delta - \tan \delta] \quad (7.28)$$

and also

$$\mathbf{R}(a+ib) \int_{\alpha} \log \frac{1+\zeta}{1-\zeta} d(\zeta^{-m}) = -2r \cos \delta \left[\psi(\frac{1}{2} - \delta/\pi) - \psi(\frac{1}{2}) \right] \quad (7.29)$$

$$\mathbf{R}(a+ib) \int_{-1}^1 \log \frac{1+\xi}{1-\xi} \overline{d\omega(\xi)} = 2r \cos \delta [\psi(\frac{1}{2} + \delta/\pi) - \psi(\frac{1}{2})] \quad (7.30)$$

$$\mathbf{R}(a+ib) \int_{\alpha} \log (1+\zeta) d(\zeta^{-m}) = r \cos \delta [\psi(\frac{1}{2}) - \psi(\frac{1}{2} - \delta/\pi)] \quad (7.31)$$

$$\mathbf{R}(a+ib) \int_{-1}^1 \log (1+\xi) \overline{d\omega(\xi)} = r \cos \delta [\psi(\frac{1}{2} + \delta/\pi) - \psi(\frac{1}{2})], \text{ where } r = |a+ib| \quad (7.32)$$

We then get

$$N_{13} = (1 + \sigma) \left[\frac{1}{4} a \sin^2 \delta (2 \cos^2 \delta - 3) - a^2 b \delta + \frac{1}{2} \sin^2 \delta \right] + \sigma \left[\frac{3}{8} b^2 \sin^2 \delta + \frac{b \sin 4\delta}{16} - \frac{1}{8} \sin^2 \delta \right], \quad (7.33)$$

$$N_{23} = (1 + \sigma) \left[ab^2 \delta - \frac{1}{4} \sin 2\delta + \frac{b \sin 2\delta \sin 4\delta}{16} + \frac{a \sin 6\delta}{32} + \frac{9b \cos^2 \delta}{16} \right] \\ + \sigma \left[-\frac{1}{3} a^2 \sin 2\delta + \frac{1}{12} \sin 2\delta + \frac{a \sin 4\delta}{16} \right] \quad (7.34)$$

Hence,

$$N_1 = (1 + \sigma) \left[-\frac{\sin^2 \delta}{15} - \frac{b \sin 2\delta}{4} + \frac{\sin 2\delta}{\pi} \left\{ 3\psi\left(\frac{1}{2} + \delta/\pi\right) - \frac{8}{5}\psi\left(\frac{1}{2} + m\right) \right. \right. \\ \left. \left. + \frac{1}{5}\psi\left(\frac{1}{2} + 3\delta/\pi\right) - 2\psi\left(\frac{1}{2}\right) \right\} \right] + \sigma \left[\frac{1}{5} \sin^2 \delta + (b/2\pi) \left\{ m\psi'\left(\frac{1}{2} + m\right) - 2\psi\left(\frac{1}{2} + m\right) + 2\psi\left(\frac{1}{2}\right) \right\} \right. \\ \left. + \frac{\sin 2\delta}{\pi} \left\{ -\frac{1}{5}\psi\left(\frac{1}{2} + 3\delta/\pi\right) - \frac{1}{3}\psi\left(\frac{1}{2} + \delta/\pi\right) + \frac{8}{15}\psi\left(\frac{1}{2} + m\right) \right\} \right], \quad (7.35)$$

$$N_2 = (1 + \sigma) \left[\frac{\sin 2\delta}{30} + \frac{1}{4} a \sin 2\delta - \frac{\cos^2 \delta}{6\pi} \left\{ \frac{4}{3}\psi\left(\frac{1}{2} + 3\delta/\pi\right) - \frac{2}{3}\psi\left(\frac{1}{2} + m\right) \right. \right. \\ \left. \left. + 12\psi\left(\frac{1}{2} + \delta/\pi\right) - 8\psi\left(\frac{1}{2}\right) \right\} \right] + \sigma \left[-\frac{\sin 2\delta}{10} - (a/\pi) \left\{ (\delta/\pi)\psi'\left(\frac{1}{2} + m\right) - \psi\left(\frac{1}{2} + m\right) + \psi\left(\frac{1}{2}\right) \right\} \right. \\ \left. + \frac{\cos^2 \delta}{2\pi} \left\{ \frac{4}{3}\psi\left(\frac{1}{2} + 3\delta/\pi\right) - \frac{2}{3}\psi\left(\frac{1}{2} + m\right) + \frac{4}{3}\psi\left(\frac{1}{2} + \delta/\pi\right) \right\} \right] \quad (7.36)$$

These values of the flexural couples are valid for the exceptional values $\pi/6, \pi/4, \pi/3, 3\pi/4$.

8. It remains to find the co-ordinates of the centre of flexure. x_f , the x -coordinate of the centre of flexure referred to parallel axes through the C.G. is given by Ghosh (1947a, p. 7)

$$x_f = \frac{I_{11}N_2 - I_{12}N_1}{2(1 + \sigma)(I_{11}I_{22} - I_{12}^2)} \quad (9.1)$$

and since the cross-section is symmetrical about the bisector of the angle 2δ between the bounding radii, the centre of flexure is on the bisector at a distance r_f from the centre given by

$$r_f = (a + x_f) \sec \delta \quad (9.2)$$

where, referred to parallel axes through the C.G.,

$$I_{11} = \iint x^2 dx dy = \frac{1}{4} \delta + \frac{\sin 4\delta}{16} - \delta a^2 \quad (9.3)$$

$$I_{12} = \iint xy dx dy = \frac{1}{8} \sin^2 2\delta - \delta ab \quad (9.4)$$

$$I_{22} = \iint y^2 dx dy = \frac{1}{4} \delta - \frac{\sin 4\delta}{16} - \delta b^2 \quad (9.5)$$

integrals being taken over the area of the cross-section. From (7.35), (7.36) and (9.1) we get

$$\begin{aligned} \frac{1}{2}(1+\sigma)(2\delta - \sin 2\delta)x_f = & (1+\sigma)\left[\left(\frac{1}{2}a + \frac{1}{3}\pi\right)\sin 2\delta - \frac{2\cos^2 \delta}{\pi}\left\{\psi\left(\frac{1}{2} + \delta/\pi\right)\right.\right. \\ & - \frac{2}{3}\psi\left(\frac{1}{2} + 2\delta/\pi\right) + \frac{1}{18}\psi\left(\frac{1}{2} + 3\delta/\pi\right) - \frac{2}{3}\psi\left(\frac{1}{2}\right)\}\Big] + \sigma\left[-\frac{1}{3}\sin \delta \cos \delta - \frac{2\cos^2 \delta}{\pi}\left\{-\frac{1}{3}\psi\left(\frac{1}{2} + 3\delta/\pi\right)\right.\right. \\ & \left. + \frac{2}{18}\psi\left(\frac{1}{2} + 2\delta/\pi\right) - \frac{1}{3}\psi\left(\frac{1}{2} + \delta/\pi\right)\right\} - (a/\pi)\{(\delta/\pi)\psi'(\frac{1}{2} + 2\delta/\pi) - \psi(\frac{1}{2} + 2\delta/\pi) + \psi(\frac{1}{2})\}] \quad (9.6) \end{aligned}$$

In conclusion I express my gratefulness to Dr. S. Ghosh for his helpful suggestions all throughout the work.

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LINEARIZED TRANS-SONIC CONICAL FLOWS

BY

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Summary. The problem of flow of a compressible liquid past a cone with axis in the direction of the undisturbed stream has been solved by direct integration of the linearized equation both for super-sonic and for sub-sonic flows. This has been possible by means of substitutions suitable for conical boundaries. The pressure-coefficient in both the cases has been tabulated for different values of the angle of the cone and of the Mach number.

1. Formulation. The problem of the flow of a compressible liquid around a body of revolution has been solved on the assumption of the existence of a potential function which satisfies a linearized differential equation. For a cone with axis parallel to the undisturbed flow, take the x -axis along the axis of the cone and the y -axis normal to the x -axis in the meridian plane in the direction of the radius of every circular cross-section of the cone. If V be the velocity of the undisturbed stream in the direction of x -axis, the velocity potential function ϕ satisfies the differential equation,

$$(1-M^2) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{1}{y} \frac{\partial \phi}{\partial y} = 0, \quad (1)$$

where $M = v/a$ is the Mach number of the undisturbed stream and a the local speed of sound.

The components of velocity are

$$\left. \begin{aligned} \frac{\partial \phi}{\partial x} &= V + u = V + \frac{\partial \phi_1}{\partial x}, \\ \frac{\partial \phi}{\partial y} &= v = \frac{\partial \phi_1}{\partial y} \end{aligned} \right\} \quad (2)$$

and

when ϕ_1 is the potential function which defines the variation of the flow generated by the presence of the body.

Evidently ϕ_1 also satisfies the equation

$$(1-M^2) \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} + \frac{1}{y} \frac{\partial \phi_1}{\partial y} = 0. \quad (3)$$

The problem has been solved (Antonio Ferri, 1949) from the potential of a source (or sink) distribution along the axis of the body, the distribution depending on the shape of the body. In this paper, the problem has been solved by direct integration of the equation (3), by means of substitutions suitable to a conical body.

Take the origin at the vertex of the cone so that at any point on the surface of the

cone $y/x = \text{constant} = \tan \alpha$, where α is the semi-vertical angle of the cone. Since the flow must graze the surface of the cone, the boundary-condition is

$$\frac{v}{V+u} = \text{constant, when } \frac{y}{x} = \text{constant.}$$

Thus we will have to find a solution of (8) such that $\left(\frac{\partial \phi_1}{\partial y}\right) / \left(V + \frac{\partial \phi_1}{\partial x}\right)$ must be constant when $y/x = \text{constant}$. Since V is constant, this is possible only when both $\partial \phi_1 / \partial x$ and $\partial \phi_1 / \partial y$ are functions of y/x .

This is satisfied if we take

$$\phi_1 = x f(\eta), \quad (4)$$

where

$$\eta = A(x/y), \quad (5)$$

A being a constant.

2. Super-sonic flow. For super-sonic flow, $M > 1$, so putting $B^2 = M^2 - 1$, the equation (3) becomes

$$\frac{B^2 \partial^2 \phi_1}{\partial x^2} = \frac{\partial^2 \phi_1}{\partial y^2} + \frac{1}{y} \frac{\partial \phi_1}{\partial y}. \quad (6)$$

With the substitutions (4) and (5), the equation (6) becomes

$$A^2 B^2 (\eta f'' + 2f') = \eta^3 f'' + \eta^2 f',$$

where dashes denote differentiations with respect to η . Taking $A = 1/B$ so that $\eta = x/By$, we get

$$\eta f'' + 2f' = \eta^3 f'' + \eta^2 f'. \quad (7)$$

This equation can be at once integrated and the solution is

$$f = -K \left[\cosh^{-1} \eta - \frac{(\eta^2 - 1)^{1/2}}{\eta} \right], \quad (8)$$

where K is a constant, so that

$$\phi_1 = -K \left[x \cosh^{-1} \frac{x}{By} - (x^2 - B^2 y^2)^{1/2} \right]. \quad (9)$$

This is exactly the solution as obtained by the method of source-distribution (Antonio Ferri, 1949).

The boundary-condition then gives

$$\frac{K}{V} = \frac{\tan \alpha}{(\cot^2 \alpha - B^2)^{1/2} + \tan \alpha \cosh^{-1}(\cot \alpha / B)}. \quad (10)$$

In order that K may be real, $\cot \alpha$ must be greater than B .

3. Sub-sonic flow: For sub-sonic flow, $M < 1$ so putting $B^2 = 1 - M^2$, the equation (8) becomes

$$B^2 \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} + \frac{1}{y} \frac{\partial \phi_1}{\partial y} = 0 \quad (11)$$

With the substitutions (4) and (5), the equation (11) becomes

$$\eta f'' + 2f' + \eta^3 f'' + \eta^2 f' = 0, \quad (12)$$

where $\eta = x/By$.

The solution of this equation is

$$f = -K \left[\sinh^{-1} \eta - \frac{(\eta^2 + 1)^{\frac{1}{2}}}{\eta} \right], \quad (13)$$

so that

$$\phi_1 = -K [x \sinh^{-1}(x/By) - (x^2 + B^2 y^2)^{\frac{1}{2}}]. \quad (14)$$

The boundary-condition gives

$$\frac{K}{V} = \frac{\tan \alpha}{(\cot^2 \alpha + B^2)^{\frac{1}{2}} + \tan \alpha \sinh^{-1}(\cot \alpha/B)}. \quad (15)$$

The case $M = 0$ or $B = 1$ corresponds to the incompressible potential flow. Then

$$\phi_1 = -K [x \sinh^{-1}(x/y) - (x^2 + y^2)^{\frac{1}{2}}], \quad (16)$$

which satisfies the Laplace's equation.

4. Pressure-coefficient. With the hypothesis of small disturbance for which the linearized equation is true, the pressure-coefficient is given by

$$c_p = \frac{\Delta p}{\frac{1}{2} \rho V^2} = -\frac{2u}{V}. \quad (17)$$

TABLE 1

| M | $\alpha = 10^\circ$ | $\alpha = 20^\circ$ | $\alpha = 30^\circ$ |
|-----|---------------------|---------------------|---------------------|
| 0.0 | 0.136 | 0.358 | 0.552 |
| 0.4 | 0.141 | 0.371 | 0.584 |
| 0.6 | 0.149 | 0.397 | 0.628 |
| 0.8 | 0.165 | 0.445 | 0.712 |
| 1.2 | 0.166 | 0.446 | 0.734 |
| 1.5 | 0.138 | 0.365 | 0.6 |
| 2.0 | 0.115 | 0.3 | — |
| 2.5 | 0.1 | 0.26 | — |
| 3.0 | 0.092 | — | — |
| 4.0 | 0.074 | — | — |

For super-sonic flow,

$$c_p = \frac{2K}{V} \cosh^{-1} \frac{\cot \alpha}{B}, \quad (18)$$

where K/V is given by (10), and for sub-sonic flow,

$$c_p = \frac{2K}{V} \sinh^{-1} \frac{\cot \alpha}{B}, \quad (19)$$

where K/V is given by (15).

The values of c_p for different values of M and of α , given in Table 1.

For super-sonic flows, the pressure-coefficient increases as the angle of the cone increases but decreases as M increases. For sub-sonic flows, the pressure coefficient increases when either the angle or M increases.

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A TEST FOR THE CONVERGENCE OF A FOURIER SERIES

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1. The object of the present note is to prove a criterion for the convergence of a Fourier series. With the usual simplifications, which do not take away the generality of the problem, we assume $f(t)$ to be even, periodic and integrable- L in $(-\pi, \pi)$ and $f(0) = 0$ and we also assume that $A_0 = 0$, and

$$f(t) \sim \sum_1^{\infty} A_n \cos nt \quad (1.1)$$

where

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos nt \, dt \quad (1.2)$$

It is sufficient to study the convergence of (1.1)

at $t = 0$, i.e. of $\sum_1^{\infty} a_n$.

Hardy & Littlewood (Hardy and Littlewood, 1932; Zygmund, 1935) proved the following;

Theorem A. *If*

$$(i) \, f(t) = o\left\{\left(\log \frac{1}{t}\right)^{-1}\right\} \text{ and } (ii) \, a_n = O(n^{-\delta})$$

for some positive δ , then $\sum a_n = 0$.

Our object is to prove the following:

Theorem 1. *If*

$$(i) \, f(t) = o\left\{\left(\log \log \frac{1}{t}\right)^{-1}\right\} \text{ and } (ii) \, a_n = O\left\{\frac{(\log n)^{\Delta}}{n}\right\}$$

for some positive Δ however large, then $\sum a_n = 0$.

In Theorem A, the emphasis is on small δ , where as in Theorem 1, the emphasis is on large Δ .

2. **Definition:**—Let $\lambda(\omega)$ be a continuous function steadily increasing to ∞ . Then $\sum a_n$ is said to be summable $(R, \lambda(\omega), k)$, where $k > 0$ to sum S , if

$$\frac{1}{\{\lambda(\omega)\}^k} \sum_{n \leq \omega} \{\lambda(\omega) - \lambda(n)\}^k a_n \rightarrow S \text{ as } \omega \rightarrow \infty. \quad (2.1)$$

* \sum stands for \sum_1^{∞}

In order to prove Theorem 1, we first prove the following :

Theorem 1.1. *If*

$$f(t) = o \left\{ \left(\log \log \frac{1}{t} \right)^{-1} \right\}$$

then $\sum a_n$ is summable $(R, \exp(\log \omega)^\Delta, 1)$ to sum 0, where Δ is any positive number.

By (2.1) $\sum a_n$ is summable $(R, \exp(\log \omega)^\Delta, 1)$ to sum 0, if

$$\frac{1}{\exp(\log \omega)^\Delta} \sum_{n \leq \omega} \{ \exp(\log \omega)^\Delta - \exp(\log n)^\Delta \} a_n \rightarrow 0 \text{ as } \omega \rightarrow \infty \quad (2.2)$$

This is equivalent to proving

$$\int_0^\pi f(t) g(\omega, t) dt \rightarrow 0 \text{ as } \omega \rightarrow \infty \quad (2.3)$$

where

$$g(\omega, t) = \frac{1}{\exp(\log \omega)^\Delta} \sum_{n \leq \omega} \{ \exp(\log \omega)^\Delta - \exp(\log n)^\Delta \} \cos nt \quad (2.4)$$

3. We now obtain the following inequalities for $g(\omega, t)$ which will be used in course of the proof of Theorem 1.1.

$$g(\omega, t) = O(\omega) \quad (3.1)$$

$$g(\omega, t) = O(t^{-1}) \quad (3.2)$$

$$g(\omega, t) = O\left(\frac{(\log \omega)^\Delta}{\omega} t^{-1}\right) \quad (3.3)$$

Proof of (3.1).

$$\begin{aligned} |g(\omega, t)| &\leq \frac{1}{\exp(\log \omega)^\Delta} \{ \exp(\log \omega)^\Delta \sum_{n \leq \omega} 1 + \sum_{n \leq \omega} \exp(\log n)^\Delta \} \\ &= \frac{1}{\exp(\log \omega)^\Delta} O(\exp(\log \omega)^\Delta \omega) = O(\omega). \end{aligned}$$

Proof of (3.2).

$$\begin{aligned} |g(\omega, t)| &\leq \frac{1}{\exp(\log \omega)^\Delta} \{ \exp(\log \omega)^\Delta - \exp(\log 1)^\Delta \} \left| \sum_1^m \cos nt \right|, \text{ where } m = [\omega]. \\ &\leq \frac{1}{\exp(\log \omega)^\Delta} (\exp(\log \omega)^\Delta - 1) \frac{1}{\sin \frac{1}{2}t} = O(t^{-1}). \end{aligned}$$

Proof of (3.3).

By partial summation, we have

$$\begin{aligned} &\sum_1^m \{ \exp(\log \omega)^\Delta - \exp(\log n)^\Delta \} \cos nt \\ &= \sum_1^{m-1} [\{ \exp(\log \omega)^\Delta - \exp(\log n)^\Delta \} - \{ \exp(\log \omega)^\Delta - \exp(\log n+1)^\Delta \}] \sum_1^n \cos nt \\ &\quad + \{ \sum_1^m \cos nt \} \{ \exp(\log \omega)^\Delta - \exp(\log m)^\Delta \} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2 \sin \frac{1}{2} t} \sum_1^{m-1} \left\{ \frac{\Delta}{n} (\log \overline{n+\theta})^{\Delta-1} \cdot \exp (\log \overline{n+\theta})^{\Delta} \sin (n+\frac{1}{2})t \right. \\
&\quad \left. + \frac{1}{2} \sum_1^{m-1} \{ \exp (\log \overline{n+1})^{\Delta} - \exp (\log \overline{n})^{\Delta} \} + \left\{ \sum_1^m \cos nt \right\} \{ \exp (\log \overline{m+1})^{\Delta} \right. \\
&\quad \left. - \exp (\log \overline{m})^{\Delta} \} \right\}, \quad (m < \omega \leq m+1, 0 < \theta < 1) \\
&= O\left(\frac{\Delta}{\omega} (\log \omega)^{\Delta-1} t^{-2} \exp (\log \omega)^{\Delta}\right) + O(\exp (\log \omega)^{\Delta}) + O\left(\frac{(\log \omega)^{\Delta-1}}{\omega} t^{-1} \exp (\log \omega)^{\Delta}\right)
\end{aligned}$$

Hence

$$g(\omega, t) = O\left(\frac{\Delta}{\omega} (\log \omega)^{\Delta-1} t^{-2}\right)$$

4. *Proof of Theorem 1.1.* We are to prove that

$$J = \int_0^{\pi} f(t)g(\omega, t) dt = o(1) \text{ as } \omega \rightarrow \infty.$$

Write

$$J = \int_0^{\omega^{-1}} + \int_{\omega^{-1}}^{\omega^{-1}(\log \omega)^{\Delta_1}} + \int_{\omega^{-1}(\log \omega)^{\Delta_1}}^{\eta} + \int_{\eta}^{\pi} = J_1 + J_2 + J_3 + J_4,$$

where Δ_1 and η will be defined presently.

Given ε , there is an η , such that

$$|f(t)| < \varepsilon \left(\log \log \frac{1}{t} \right)^{-1}, \text{ for } 0 < t \leq \eta.$$

Also choose

$$\eta > (\log \omega)^{\Delta_1} / \omega.$$

By (3.1)*

$$|J_1| < k\varepsilon \cdot \int_0^{\omega^{-1}} \omega dt = k\varepsilon \quad (4.1)$$

By (3.2)

$$\begin{aligned}
|J_2| &< k\varepsilon \int_{\omega^{-1}}^{\omega^{-1}(\log \omega)^{\Delta_1}} \frac{dt}{\log \log 1/t} \\
&= k\varepsilon \left[\int_{\omega^{-1}}^{\omega^{-1}(\log \omega)^{\Delta_1}} \frac{-d(\log 1/t)}{\log \log (1/t)} \right] < k\varepsilon \cdot \frac{\Delta_1 \log \log \omega}{\log (\log \omega - \Delta_1 \log \log \omega)} \\
|J_2| &\leq k\varepsilon \Delta_1
\end{aligned} \quad (4.2)$$

Hence

By (3.3)

$$\begin{aligned}
|J_3| &< k\varepsilon \int_{\omega^{-1}(\log \omega)^{\Delta_1}}^{\eta} \frac{(\log \omega)^{\Delta-1}}{\omega} \frac{dt}{t^2} \\
&= k\varepsilon \frac{(\log \omega)^{\Delta-1}}{\omega} \left[\frac{\omega}{(\log \omega)^{\Delta_1}} - \frac{1}{\eta} \right] < k\varepsilon \text{ if } \Delta_1 \geq \Delta - 1
\end{aligned} \quad (4.3)$$

* k stands for a constant independent of ω and t and is not necessarily the same at each occurrence

Using (8.8) again, we have ‡

$$|J_4| < k \frac{(\log \omega)^{\Delta-1}}{\omega} \int_{\eta}^{\omega} \frac{|f(t)|}{t^2} dt = k \frac{(\log \omega)^{\Delta-1}}{\omega} A$$

Thus $J_4 = o(1)$ as $\omega \rightarrow \infty$ (4.4)

Finally by (4.1), (4.2), (4.8) and (4.4) $J = o(1)$, which completes the proof of Theorem 1.1.

5. Combining Theorem 1.1 with the case $\lambda(\omega) = \exp(\log \omega)^{\Delta}$, $k = 1$, of Theorem B (Hardy and Riesz, 1914), below, Theorem 1 follows.

Theorem B. If

$$(i) \sum a_n \text{ is summable } (R, \lambda, k), k > 0, \text{ and } (ii) a_n = O \left\{ \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \right\}$$

then $\sum a_n$ is convergent.

6. *Concluding Remarks.* If the hypothesis of Theorem 1.1 be replaced by

$$f(t) = O\{(\log \log 1/t)^{-1}\}$$

we can still prove that $J = o(1)$, provided that we choose $\Delta \leq 1 + \Delta_1$, where Δ_1 is arbitrarily small.

We thus have the following :

Theorem 2. If

$$(i) f(t) = O \left\{ \frac{1}{\log \log (1/t)} \right\} \text{ and } (ii) a_n = O \left[\frac{(\log n)^{\delta}}{n} \right]$$

for every positive δ , then $\sum a_n = 0$.

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‡ A in the following steps stands for the value of the integral $\int_{\eta}^{\omega} \frac{|f(t)|}{t^2} dt$ which certainly exists for a fixed η .

ON THE LIMITING LINES IN INVISCID ROTATIONAL AND VISCOUS FLOWS

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(Communicated by Prof. B. R. Seth—Received March 17, 1960)

Introduction

The problem of limiting lines in plane, axially symmetric and even three-dimensional inviscid flow was attacked and solved by several authors (some of them are cited at the end of the paper, 2, 12, 14, 15, 16, 17). In the present paper an attempt is presented to solve the problem of the existence or non-existence of a limiting line in a viscous fluid. The attack is based on the extended Tollmien's theory. The considerations, given below, refer parallelly to an inviscid rotational and to a viscous flow. It will be shown that under certain conditions less restrictive for an inviscid fluid and more restrictive for a viscous fluid, both types of flow may be treated simultaneously. The author did not attempt to present the way of construction of the solution and of the intermediary items (like characteristics, Mach lines, limiting lines, etc.). Only the proof concerning the existence theorem will be presented. This approach in the case of a viscous fluid enables one to use the characteristics of the reduced order equation (second order) instead of the singularities of the original equation (third order). Thus, although one is not able to find the singularities in a viscous fluid vector field of flow, one may extend the considerations referring to a limiting line to this type of flow. A brief representation of the hypothesis of a shock in viscous gases closes the paper.

I. General Laws and Basic Equations

I.1. The Equation of Motion. For a viscous fluid the equation of motion is:

$$\frac{d\mathbf{V}}{dt} = \mathbf{V}_t + \text{grad} \left(\frac{1}{2} \mathbf{V}^2 \right) - \mathbf{V} \times \boldsymbol{\omega} = \bar{\rho}^{-1} (\mathbf{F} - \text{grad } \bar{p} + \mathbf{P}), \quad (1)$$

where \mathbf{F} denotes the external forces, $\boldsymbol{\omega}$ the curl \mathbf{V} , $\bar{\rho}$ the density, \mathbf{P} the forces due to viscosity. The effect of the gravitation on the compressibility phenomena may be included into the value for the external forces. It is possible to transform equation (1) into other forms. The specific enthalpy " h " is defined by

$$h = U + \bar{\rho}^{-1} \bar{p} = J c_p \bar{T}, \quad (2)$$

where $U = J c_v \bar{T}$ denotes the specific internal energy. The specific entropy " S " is defined by the relation

$$J \bar{T} dS = dU + \bar{p} d(\bar{\rho}^{-1}) \equiv dh - \bar{\rho}^{-1} d\bar{p}, \quad (3)$$

or

$$J \bar{T} dS = \text{grad } h - \bar{\rho}^{-1} \text{grad } \bar{p}. \quad (3a)$$

By means of equation (8a) equation (1) may be put into the form:

$$\frac{d\mathbf{Y}}{dt} = \bar{\varrho}^{-1}(\mathbf{F} + \mathbf{P}) - \text{grad } h + J\bar{T} \text{ grad } S. \quad (4)$$

Applying the notion of "stagnation enthalpy", that is $h_0 = h + \frac{1}{2}\mathbf{Y}^2$, one obtains from (4):

$$\mathbf{Y}_t + \omega \times \mathbf{Y} = \bar{\varrho}^{-1}(\mathbf{F} + \mathbf{P}) + J\bar{T} \text{ grad } S - \text{grad } h_0 = \mathbf{A}. \quad (5)$$

For a steady flow ($\mathbf{Y}_t = 0$) equation (5) is the "vortex equation". Differentiating the equation of "stagnation enthalpy" with respect to time, keeping in mind that $d\bar{p}/dt = \bar{p}_t + \mathbf{Y} \cdot \text{grad } \bar{p}$, and using equation (3) one obtains:

$$\frac{dh_0}{dt} = J\bar{T} \left(\frac{dS}{dt} \right) + \bar{\varrho}^{-1} \bar{p}_t + \mathbf{Y} \cdot \left[\bar{\varrho}^{-1} \text{grad } \bar{p} + \left(\frac{d\mathbf{Y}}{dt} \right) \right]. \quad (6)$$

Replacing the expression inside the bracket of the last term by a value obtained from equation (1) one obtains (in accordance with Ref. 18 for $\mu = 0$):

$$\frac{dh_0}{dt} = J\bar{T}(dS/dt) + \bar{\varrho}^{-1} \bar{p}_t + \mathbf{Y} \cdot [\bar{\varrho}^{-1}(\mathbf{F} + \mathbf{P})]. \quad (6a)$$

I.2. Equations of Continuity and State

$$\bar{\varrho}_t + \text{div}(\bar{\varrho}\mathbf{Y}) = 0, \quad \text{or} \quad \text{div } \mathbf{Y} = (-\bar{\varrho}^{-1}) \left(\frac{d\bar{\varrho}}{dt} \right), \quad (7)$$

$$\bar{p} = \bar{R}\bar{\varrho}\bar{T}. \quad (8)$$

I.3. Equation of Energy. Equation of energy may be written in the form:

$$Jc_v \bar{\varrho} D\bar{T} + \bar{p} \text{div } \mathbf{Y} = J \text{div}(k \text{ grad } \bar{T}) + \Phi, \quad (9)$$

$$D\bar{T} = d\bar{T}/dt = \bar{T}_t + \mathbf{Y} \cdot \text{grad } \bar{T}, \quad (9a)$$

where "k" denotes the coefficient of heat conductivity and Φ is the dissipation function.

Using the second of equations (7) and substituting $\bar{p}\bar{\varrho}^{-2}(d\bar{\varrho}/dt) = -\bar{p}(d/dt)(\bar{\varrho}^{-1})$, one may transform equation (9) into the form:

$$\bar{\varrho}[Jc_v D\bar{T} + \bar{p}(d/dt)(\bar{\varrho}^{-1})] = J \text{div}(k \text{ grad } \bar{T}) + \Phi. \quad (9b)$$

Setting $dU = Jc_v d\bar{T}$ into equation (8), one gets from equation (9b):

$$J\bar{\varrho}\bar{T} \left(\frac{dS}{dt} \right) = J \text{div}(k \text{ grad } \bar{T}) + \Phi. \quad (10)$$

The scalar product of equation (5) by \mathbf{Y} in the case of a steady flow gives

$$\mathbf{Y} \cdot \mathbf{A} = 0, \quad (11)$$

which means that the vector \mathbf{A} is perpendicular to \mathbf{Y} (or to a stream-line).

In case of a compressible, inviscid, isentropic, rotational flow, one has to substitute in the equations derived above $\mathbf{P} = k = \Phi = 0$. Various vortex theorems are cited in [6, 13, 18].

I.4. Density. As is known from the theory of sound, the effect of viscosity and

heat conductivity on the velocity of sound in a real gas may be neglected up to the first order of a quantity which includes the wave length (8, 9). Hence the local velocity of sound is the same as if the adiabatic flow conditions were preserved. One may write

$$V_s^2 = d\bar{p}/d\bar{\rho} = \gamma\bar{p}\bar{\rho}^{-1} = \gamma\bar{R}\bar{T}, \quad (12)$$

where V_s denotes the sound velocity, and γ the ratio of specific heats. Dividing equation (8) by \bar{T} , setting $U = Jc_p d\bar{T}$, and then using the concept of stagnation enthalpy jointly with equation (2) one obtains the formula for the density (it differs from that in Ref. 18):

$$\bar{\rho} = \bar{\rho}_0 f^{-1} (\bar{T}/\bar{T}_0)^{1/(\gamma-1)} = \bar{\rho}_0 f^{-1} (V_s/V_{s0})^{2/(\gamma-1)} = \bar{\rho}_0 f^{-1} [h_0 - \frac{1}{2} V_s^2]^{1/(\gamma-1)} / (Jc_p \bar{T}_0)^{1/(1-\gamma)}, \quad (13)$$

$$f = f(S) = \exp [J(S - S_0) \bar{R}^{-1}],$$

the constant of integration being determined from the conventional initial conditions of rest ($p_0, \bar{\rho}_0, \bar{T}_0, S_0, V_{s0}$) and being equal to $JS_0 - \bar{R}(\gamma-1)^{-1} \log [\bar{p}_0 (\bar{R}\bar{\rho}_0 \gamma)^{-1}]$.

In order to operate with dimensionless quantities, introduce the notion of a limiting velocity V_l which will be defined as the velocity of expansion into a vacuum in isentropic conditions:

$$\frac{1}{2} V_l^2 = \gamma(\gamma-1)^{-1} (\bar{p}_0/\bar{\rho}_0) = Jc_p \bar{T}_0. \quad (14)$$

The following dimensionless quantities are introduced:

$$\mathbf{V}/V_l = \mathbf{q}, \quad \bar{\rho}/\bar{\rho}_0 = \rho, \quad h_0/V_l^2 = h_{01}, \quad \bar{T}/\bar{T}_0 = T, \quad V_s/V_l = c,$$

$$\bar{p}/\bar{p}_0 = p, \quad (Jc_p \bar{T})/V_l^2 = \frac{1}{2}, \quad T = c^2/c_0^2.$$

The coordinates also are assumed to be dimensionless. The temperature \bar{T}_0 may vary in the domain of the flow (non-iso-energetic gas).

1.5. Vortex Theorem. The dimensionless density is of the form:

$$\rho = f^{-1} \{2[h_{01} - \frac{1}{2} c^2]\}^{1/(\gamma-1)} = f^{-1} (c/c_0)^{2/(\gamma-1)}, \quad (15)$$

or

$$\rho = f^{-1} T^{1/(\gamma-1)}, \quad (15a)$$

with $h/(V_l^2) = \frac{1}{2} T$.

Equation (5) for a steady flow may be written in a dimensionless form:

$$\boldsymbol{\omega} \times \mathbf{q} = \rho^{-1} (\mathbf{F} + \mathbf{P} - \text{grad } p) - \text{grad } (\frac{1}{2} q^2), \quad (16)$$

where \mathbf{F} and \mathbf{P} are dimensionless.

Assume that the functions f and T are certain functions of a stream function ψ and position, i.e., $f = f(\psi, x, y)$ and $T = T(\psi, x, y)$. Hence $\text{grad } p = \text{grad } (R\rho T)$, or

$$\text{grad } p = R\{-f^{-2} T^{\gamma/(\gamma-1)} (f_\psi \text{grad } \psi + \text{grad } f) + [\gamma/(\gamma-1)] f^{-1} T^{1/(\gamma-1)} (T_\psi \text{grad } \psi + \text{grad } T)\}, \quad (17)$$

and

$$-\rho^{-1} \text{grad } p = R\{f^{-1} T (f_\psi \text{grad } \psi + \text{grad } f) + [\gamma/(\gamma-1)] (T_\psi \text{grad } \psi + \text{grad } T)\}. \quad (17a)$$

Hence

$$\begin{aligned} \boldsymbol{\omega} \times \mathbf{q} = \rho^{-1} (\mathbf{F} + \mathbf{P}) + R\{f^{-1} T (f_\psi \text{grad } \psi + \text{grad } f) \\ + \gamma(\gamma-1)^{-1} (T_\psi \text{grad } \psi + \text{grad } T)\} - \text{grad } (\frac{1}{2} q^2). \end{aligned} \quad (18)$$

This equation appears to be new. One may assume that it contains the most generalized form of Bjerknes' vortex theorem for the viscous flows, and represents an extension of Tollmien's theorem (16. Brown, p. 8).

II. Two-Dimensional Steady Flow without the External Forces

II.1. Stream-Function Equation. In this case $\mathbf{q} = \mathbf{q}(u, v)$, $u = q \cos \theta$, $v = q \sin \theta$, and the continuity equation is

$$(uq)_x + (vq)_y = 0. \quad (19)$$

Introducing the notion of a stream function ψ one obtains from equation (19):

$$\psi_x^2 + \psi_y^2 = |\text{grad } \psi|^2 = q^2 q^2. \quad (20)$$

Similarly:

$$v/u = \tan \theta = -\psi_x/\psi_y, \quad u\psi_x + v\psi_y = 0. \quad (20a)$$

Differentiate $v = -q^{-1}\psi_x$ partially with respect to x , $u = q^{-1}\psi_y$ partially with respect to y and subtract the latter from the first, which gives:

$$q(v_x - u_y) = (uq_y - vq_x) - (\psi_{xx} + \psi_{yy}). \quad (21)$$

Take gradients on both (extreme) sides of equation (20), represent gradients as the sum of two vectors (i.e., $\text{grad } q = i q_x + j q_y$), multiply the result vectorially by $\mathbf{q} = i u + j v$ and take the scalar of the final vector (which is in \mathbf{k} direction). This gives:

$$q^2(uq_y - vq_x) + qq(uq_y - vq_x) = u^2\psi_{yy} - 2uv\psi_{xy} + v^2\psi_{xx}. \quad (22)$$

In equation (3) replace all the quantities by dimensionless ones, the factor dp by $(dp/dq)dq$ and the ratio dp/dq by c^2 . Calculate from that relation the differential dq , which may be used to find the values of q_x and q_y . After a few transformations obtain:

$$\begin{aligned} eqq_x &= qh_{01x} - \frac{1}{2}qTc_p^{-1}S_x - c^2q_x = A_1 - c^2q_x, \\ eqq_y &= qh_{01y} - \frac{1}{2}qTc_p^{-1}S_y - c^2q_y = A_2 - c^2q_y. \end{aligned} \quad (23)$$

Denote by s the running coordinate along a streamline, by n the normal to a streamline. Since along a streamline $\psi = \text{const.}$, $\text{grad } \psi$ is directed perpendicularly to a streamline, and one has from equation (20):

$$d\psi/dn = q. \quad (24)$$

By taking absolute values on both sides of equation (5) and keeping in mind that the vector \mathbf{A} has no component along a streamline one gets the following result after division by equation (24) (with dimensionless quantities):

$$\omega = v_x - u_y = q\left[\frac{1}{2}Tc_p^{-1}(dS/d\psi) - (dh_{01}/d\psi)\right] + \xi(dn/d\psi), \quad (25)$$

where the symbol ξ represents the sum of all the terms due to viscosity.

Insert equation (23) into equation (22). Put the resultant equation into equation (21) multiplied by $(q^2 - c^2)$. Using equation (25) one obtains in the final result the stream-function equation:

$$(c^2 - u^2)\psi_{xx} - 2uv\psi_{xy} + (c^2 - v^2)\psi_{yy} = \sum C_i, \quad (26)$$

$$C_1 = (q^2 - c^2)q\omega, \quad C_2 = A_2u, \quad C_3 = -A_1v. \quad (26a)$$

II.2. "Generalized" Potential Equation. The concept of a function analogous to the velocity potential in a potential flow will be introduced. Namely, the equation $v_x - u_y = \omega$, where ω denotes vorticity, is fulfilled if one puts $u = \Phi_x + g$, $v = \Phi_y + g$, where the function g is defined by the partial differential equation of the form $g_x - g_y = \omega$, with the condition that for an irrotational flow ($\omega = 0$) the trivial solution of the remaining homogeneous equation, i.e., $g = 0$ must be assumed. Inserting the values for u and v , and the values for e_x and e_y , calculated in II.1., into the continuity equation $\text{div}(\rho \mathbf{q}) = 0$ with dimensionless quantities one obtains the "generalized" potential equation:

$$(c^2 - u^2)\Phi_{xx} - 2uv\Phi_{xy} + (c^2 - v^2)\Phi_{yy} = g_x\Phi_x^2 + (g_x + g_y)\Phi_x\Phi_y + g_y\Phi_y^2 + A\Phi_x + B\Phi_y - C. \quad (27)$$

The used symbols denote:

$$A = (3g_x + g_y)g - A_1 e^{-1}, \quad B = (g_x + 3g_y)g - A_2 e^{-1},$$

$$C = (c^2 - 2g^2)(g_x + g_y) + e^{-1}g(A_1 + A_2). \quad (27a)$$

III. Hodograph Transformations

III.1. Inviscid, Isentropic, Rotational Flow. In this case one has from equations (6a) and (10) along a streamline with $S = S(s, t)$ and $h_0 = h_0(s, t)$:

$$h_{0s} = S_s = 0.$$

Hence both the entropy and the stagnation enthalpy are constant along each streamline. The velocity of sound is given by the formula $V_i^2 = \frac{1}{2}(\gamma - 1)(V_i^2 - V^2)$ and h_{01} is equal to $\frac{1}{2}$ in the entire domain. Following Crocco, in this case one may easily derive the so-called "modified" continuity equation in the form:

$$\text{div}[\rho(1 - q^2)^{1/(\gamma-1)}] = 0, \quad (28)$$

which is equivalent to the original one. Similarly the "modified" density has the form:

$$\rho_1 = (1 - q^2)^{1/(\gamma-1)}. \quad (28a)$$

Equation (28a) leads to the expression $\rho_1 q q_x = -c^2 \rho_{1x}$, which must be used instead of equation (23) in the derivation of the stream-function equation. Transforming the term $\frac{1}{2} T c_p^{-1}$ into the expression $J(\gamma - 1)(1 - q^2)(2\gamma R)^{-1}$ and using equation (28a) one obtains on the right hand side of (26) the term $(q^2 - c^2)(1 - q^2)^{(\gamma+1)/(\gamma-1)} f$, $f = f(\psi) = J(\gamma - 1)(2\gamma R)^{-1} \times (dS/d\psi)$. This is exactly Crocco's equation [8] which although derived by Crocco primarily for a rotational flow of uniform stagnation enthalpy is valid for a rotational flow having non-uniform stagnation enthalpy if dimensionless quantities with respect to V_i are used as shown by Prim [11]. Hence the quantities h_0 and S may vary from streamline to streamline (non-iso-energetic or non-homentopic flow). The quantity h_{01} is equal to $\frac{1}{2}$ because $h_c = \frac{1}{2} V_i^2$ and $h_0/V_i^2 = \frac{1}{2}$. The "generalized" potential equation will be also simplified. The application of the "modified" density leads to a modification of the coefficients A , B , and C in equation (27). The second term in each of these three coefficients is equal to zero. This may be easily verified by means of equation (28a). For an irrotational flow equation (27) transforms into the well-known potential equation.

To solve the equation $g_x - g_y = \omega$, let it remember that the quantity ω is equal to (see Crocco, Tollmien and others):

$$\omega = (2\gamma)^{-1}(1 - q^2)^{\gamma/(\gamma-1)}(d/d\psi)[\log f(\psi)] = \omega[g(x, y), f(\psi(x, y))].$$

A solution of this equation by means of elementary methods is:

$$f(c_1, c_2) = 0, \quad x + y = c_1,$$

or

$$g - \frac{1}{2} \left[\int \omega(c_1, x) dx - \int \omega(c_1, y) dy \right] = c_2,$$

$$g = x + y + \frac{1}{2} \left[\int \omega(c_1, x) dx - \int \omega(c_1, y) dy \right].$$

Thus it is obvious that in the case of an inviscid, isentropic, rotational flow the right-hand sides of equations (26) and (27) are functions of Φ_x , Φ_y , $q(x, y)$, $f(\psi)$, only (the derivatives g_x and g_y are functions of ω).

III.2. Viscous Flow. A vector field of flow of a viscous fluid may be considered to be composed of a velocity potential vector field and a superimposed perturbation velocity vector field. The latter field is due to viscous stresses, external forces, energy exchange by means of viscosity, heat conductivity, etc. It should fulfill certain conditions such as the viscous stresses be proportional to second derivatives of the velocity components, etc. Above, the concept of a generalized velocity potential was introduced involving the vorticity ω through the function g . In the most general case of a viscous fluid the vorticity is a complicated function of few parameters like entropy, total enthalpy, viscous stresses, etc. But below it will be shown that in order to prove a theorem referring to a limiting line, one may introduce a few simplifications. Assume a quasi-linear third order partial differential equation of the form

$$a_0 \Phi_{xxx} + b_0 \Phi_{xyy} + \dots + a_1 \Phi_{xx} + b_1 \Phi_{xy} + c_1 \Phi_{yy} + a_2 \Phi_x + b_2 \Phi_y + a_3 = 0,$$

where $a_1 = a_1(\Phi_x, \Phi_y, \Phi, x, y)$, b_1 and c_1 similarly, $a_2 = a_2(\Phi, x, y)$, b_2 similarly and $a_3 = a_3(\Phi, x, y)$. Assume that $\Phi = \Phi(x, y)$ is a solution of the given equation, satisfying the prescribed boundary conditions. Consequently, the terms involving third partial derivatives of Φ may be assumed to be a certain function of Φ , x and y , equal to $F_1(\Phi, x, y)$, say. So obtained equation of the reduced order

$$a_1 \Phi_{xx} + b_1 \Phi_{xy} + c_1 \Phi_{yy} + a_2 \Phi_x + b_2 \Phi_y + a_3 + F_1 = 0,$$

is another representation of the original equation. This is a quasilinear second order differential equation and in case it possesses characteristics, they lie on the integral surface $\Phi = \Phi(x, y)$ of the original equation. Those characteristics will be uniquely determined by means of the coefficients a_1 , b_1 and c_1 which are functions of Φ (solution of the original equation). Of course, the integral surface of the original equation, Φ , possesses another characteristic curve, if any, corresponding to the original third order differential equation. Let it omit the discussion on the mutual relationship between those two families of characteristics, of the original and of the reduced order equations. Below, it will be shown, that there is possible to prove certain theorems on the limiting

line taking into account the equation of the reduced order only and neglecting the characteristic curves corresponding to the equation of higher order. Although, taking into account the equation of a reduced order, one is not able to outline the way of construction the characteristics, Mach lines, limiting lines, etc., one is able to extend certain proofs to the range of viscous fluid. From the structure of the procedure it is obvious that this extension is valid for any type of flow, even if the corresponding equation contains derivatives of m -th order, $m > 3$, provided the equation of the reduced order possesses the characteristic curves whose equation can be put down.

Following the procedure, outlined above, assume that all the function $\Phi(x, y)$, $T = T(x, y)$, $e = e(x, y)$, $S = S(x, y)$, etc., are real solutions of the given equation satisfying the prescribed boundary conditions. Consequently, such quantities like ω , e , A_1 , A_2 , g , etc., are assumed to be functions of Φ , x and y only. This means that the right-hand sides of equations (26) and (27) are functions of Φ , Φ_x , Φ_y , x , y only and the same procedure may be applied to both types of flow: rotational, inviscid, and viscous, in the latter case Φ being a solution of third order equation.

III.3. "Generalized" Molenbroek-Chaplygin Equation. Following the Molenbroek-Chaplygin idea of the hodograph method, the variables Φ and ψ will be represented as functions of q and θ . The relations between the previous (x, y) and new coordinates (q, θ) is easily found, namely

$$d\Phi = \Phi_x dx + \Phi_y dy = (u - g)dx + (v - g)dy, \quad (29a)$$

$$d\psi = \psi_x dx + \psi_y dy = e(udy - vdx). \quad (29b)$$

Solving these equations for dx and dy gives:

$$dx = \alpha \{ \cos \theta d\Phi - e^{-1} [\sin \theta - gq^{-1}] d\psi \}, \quad (30a)$$

$$dy = \alpha \{ \sin \theta d\Phi + e^{-1} [\cos \theta - gq^{-1}] d\psi \}, \quad (30b)$$

$$\alpha = \alpha(q, \theta, g) = [q - g(\sin \theta + \cos \theta)]^{-1}, \quad (30c)$$

where the relations $u = q \cos \theta$, $v = q \sin \theta$ were used. Let

$$\Phi = \Phi(q, \theta), \quad \psi = \psi(q, \theta), \quad g = g(q, \theta), \quad d\Phi = \Phi_q dq + \Phi_\theta d\theta, \text{ etc.}$$

Inserting those values into equations (30) one obtains:

$$dx = \alpha \{ [\cos \theta \cdot \Phi_q - D_1 \psi_q] dq + [\cos \theta \cdot \Phi_\theta - D_1 \psi_\theta] d\theta \}, \quad (31a)$$

$$dy = \alpha \{ [\sin \theta \cdot \Phi_q + D_2 \psi_q] dq + [\sin \theta \cdot \Phi_\theta + D_2 \psi_\theta] d\theta \}, \quad (31b)$$

$$D_1 = e^{-1} [\sin \theta - gq^{-1}], \quad D_2 = e^{-1} [\cos \theta - gq^{-1}]. \quad (31c)$$

It does not make any difference in subsequent considerations whether one uses the dimensionless density e or dimensionless quantity e_1 . Since x and y are determined by equations (31), one must have the following relations from the integrability conditions for a line integral:

$$\{x[\cos \theta \cdot \Phi_q - D_1 \psi_q]\}_\theta = \{\alpha[\cos \theta \cdot \Phi_\theta - D_1 \psi_\theta]\}_q, \quad (32a)$$

$$\{x[\sin \theta \cdot \Phi_q + D_2 \psi_q]\}_\theta = \{\alpha[\sin \theta \cdot \Phi_\theta + D_2 \psi_\theta]\}_q. \quad (32b)$$

After performing the indicated differentiation, the terms containing the higher partial derivatives cancel and the remaining expressions represent a set of two equations linear in Φ_q , Φ_θ , ψ_q , ψ_θ . Calculate Φ_q and Φ_θ :

$$\Phi_\theta = a_1\psi_\theta - a_2\psi_q, \quad (83a)$$

$$\Phi_q = b_1\psi_\theta - b_2\psi_q, \quad (83b)$$

where

$$a_1 = \{(\alpha q^{-1})_q \alpha_\theta - D_3 D_5\}(\alpha \alpha_q)^{-1}, \quad (84a)$$

$$a_2 = \{(\alpha q^{-1})_\theta + \alpha^2 q^{-1} - D_4 D_5\}(\alpha \alpha_q)^{-1}, \quad (84b)$$

$$b_1 = \{(\alpha q^{-1})_q - D_3 D_5\} \alpha^{-1}, \quad (84c)$$

$$b_2 = \{(\alpha q^{-1})_\theta - D_4 D_5\} \alpha^{-1}, \quad (84d)$$

$$D_3 = [\alpha(qq)^{-1}g]_q, \quad D_4 = [\alpha(qq)^{-1}g]_\theta,$$

$$D_5 = [\alpha(\sin \theta + \cos \theta)]_\theta, \quad D_6 = \sin \theta + \cos \theta.$$

Equations (83) transform into Tollmien's equations for an irrotational flow and in general they correspond to the Cauchy-Riemann equations for an incompressible flow.

Insert equations (83) into (81) to eliminate the function Φ :

$$dx = \alpha\{[(c_1 - b_2 \cos \theta)\psi_q + b_1 \cos \theta \psi_\theta]dq + [(-a_2 \cos \theta \psi_q) + (c_1 + a_1 \cos \theta)\psi_\theta]d\theta\}, \quad (85a)$$

$$dy = \alpha\{[(d_1 - b_2 \sin \theta)\psi_q + b_1 \sin \theta \psi_\theta]dq + [(-a_2 \sin \theta \psi_q) + (d_1 + a_1 \sin \theta)\psi_\theta]d\theta\}, \quad (85b)$$

where

$$c_1 = c_1(q, \theta, g) = gq^{-1} - \sin \theta,$$

$$d_1 = d_1(q, \theta, g) = -gq^{-1} + \cos \theta.$$

Differentiate (83a) with respect to q , and (83b) with respect to θ . Comparing the partial derivatives $\Phi_{q\theta} = \Phi_{\theta q}$, gives the result in the form:

$$a_2\psi_{qq} - (a_1 + b_2)\psi_{q\theta} + b_1\psi_{\theta\theta} + [a_{2q} - b_{2\theta}]\psi_q - [a_{1q} - b_{1\theta}]\psi_\theta = 0. \quad (86)$$

This hodograph equation corresponds to the well-known "Molenbroek-Chaplygin" equation in i^3 -flow [16]. The coefficients a_1 , a_2 , b_1 , b_2 may be assumed to be functions of q and θ only (see VIII).

III.4. Characteristic Curves. The characteristic curves of both the equations (26) and (27) satisfy in the (x, y) plane the relation

$$(c^2 - u^2)dy^2 + 2uvdx dy + (c^2 - v^2)dx^2 = 0, \quad (87)$$

or

$$c^2(dx^2 + dy^2) = (udy - vdx)^2. \quad (87a)$$

In a viscous fluid u and v correspond to a solution of the third order equation.

These characteristic curves are real in the supersonic region, i.e., for $q > c$. If θ_1 denotes the angle of the inclination of a stream-line to the x -axis at a point in the (x, y) plane, and θ_2 the angle of the inclination of a characteristic curve to the x -axis at the same point, then $\sin \theta_1 = v/q$ and θ_2 is given by equation (37). From these relations there follows, as can easily be verified, that $\sin(\theta_1 - \theta_2) = \sin \varepsilon = \pm c/q = \pm M^{-1}$, where $M = q/c$ denotes the "Mach Number." In other words $\varepsilon = \arcsin(M^{-1})$ and the characteristic curves coincide with the Mach waves. There is no difference in this respect between viscous, rotational and irrotational flows.

The "characteristic hodograph" of equation (36) (i.e., the characteristic curve in the hodograph plane) fulfills the relation:

$$a_2 d\theta^2 + (a_1 + b_2) d\theta dq + b_1 dq^2 = 0. \quad (38)$$

The characteristic hodographs correspond to Mach waves in the physical plane. The point $q = c$ should be excluded (similarly, as in an irrotational flow). For $q = c$ equation (37) gives a degenerated result $\bar{dy}/dx = -u/v$. It was not possible to derive from (38) the conclusions concerning the range of velocities for which $d\theta/dq$ takes real values. It will be shown below that eq (38) may be obtained directly from eq. (37). Since the real values of dy/dx are obtainable from (37) for $q > c$, one is justified in assuming that the real values of $d\theta/dq$ from (38) are also only for $q > c$.

To obtain the real values of the expression $d\theta/dq$ the discriminant must be less than or equal to zero, that is

$$4a_2b_1 - (a_1 + b_2)^2 \leq 0. \quad (39)$$

Solving equation (38) for $d\theta/dq$ and integrating one obtains:

$$\theta = E_1 \pm E_2, \quad (40)$$

where

$$E_1 = -\frac{1}{2} \int (a_1 + b_2) a_2^{-1} dq, \quad (40a)$$

$$E_2 = \frac{1}{2} \int a_2^{-1} [(a_1 + b_2)^2 - 4a_2b_1]^{1/2} dq. \quad (40b)$$

The two families, of the characteristic hodographs are given by the relations:

$$\xi = \theta - E_1 + E_2 = \text{const.}, \quad (41a)$$

$$\eta = \theta - E_1 - E_2 = \text{const.} \quad (41b)$$

Let: $\xi = \xi(q, \theta) = \text{const}$, $\eta = \eta(q, \theta) = \text{const}$, or

$$\xi_q dq + \xi_\theta d\theta = 0, \quad d\theta = -(\xi_q/\xi_\theta) dq = -\lambda_1 dq, \quad d\theta = -(\eta_q/\eta_\theta) dq = -\lambda_2 dq, \quad (42a)$$

which changes equation (38) into the form:

$$a_2 \lambda_i^2 - (a_1 + b_2) \lambda_i + b_1 = 0, \quad i = 1, 2. \quad (42b)$$

The canonical forms of the hodograph equation (38) will now be derived. Set:

$$\psi = \psi(\xi, \eta), \quad \psi_\eta = \psi_\xi \xi_\eta + \psi_\eta \eta_\eta, \text{ etc.}$$

and derive the characteristic quadratic forms $\bar{\alpha}$, $\bar{\gamma}$ and the characteristic polar form $\bar{\beta}$ associated with the given differential equation:

$$\begin{aligned} \bar{\alpha} &= a_2 \xi^2 - (a_1 + b_2) \xi_q \xi_\theta + b_1 \xi_\theta^2, \\ \bar{\gamma} &= a_2 \eta^2 - (a_1 + b_2) \eta_q \eta_\theta + b_1 \eta_\theta^2, \end{aligned} \quad (43)$$

$$\bar{\beta} = 2a_2 \xi_q \eta_q - (a_1 + b_2) (\xi_q \eta_\theta + \xi_\theta \eta_q) + 2b_1 \xi_\theta \eta_\theta.$$

In the case of a hyperbolic equation, i.e., $a_2 b_1 - (\frac{1}{2}(a_1 + b_2)^2 < 0$, one has $\bar{\alpha} = \bar{\gamma} = 0$, and the canonical form is:

$$\psi_{\xi\eta} + (\bar{\epsilon}_1/\bar{\beta}) \psi_\xi + (\bar{\epsilon}_2/\bar{\beta}) \psi_\eta = 0, \quad (44)$$

where

$$\bar{e}_i = \{a_2 s_{i\eta\eta} - (a_1 + b_2) s_{i\eta\theta} + b_1 s_{i\theta\theta} + (a_{2q} - b_{2\theta}) s_{i\eta} - (a_{1q} - b_{1\theta}) s_{i\theta}\},$$

$$i = 1, 2, \quad s_1 = \xi, \quad s_2 = \eta.$$

In the parabolic case $a_2 b_1 - \frac{1}{4}(a_1 + b_2)^2 = 0$, $\bar{\alpha} = \bar{\beta} = 0$, $\bar{\gamma} \neq 0$, and the canonical form is

$$\psi_{\eta\eta} + (\bar{e}_1/\bar{\gamma})\psi_\xi + (\bar{e}_2/\bar{\gamma})\psi_\eta = 0. \quad (44a)$$

The elliptic case is excluded. In an irrotational flow only the hyperbolic type may be taken into account

III.3. The Jacobian $\partial(\Phi, \psi)/\partial(x, y)$. Consider equations (29) as a set of two equations in dx and dy . The Jacobian Δ is equal to $[q^2 - g(u+v)]$ and cannot vanish identically in the entire flow domains in both the physical and hodograph planes, since the identical vanishing of this determinant in certain domains would mean that Φ and ψ are functionally related, that is, $\Phi = \Phi(\psi)$, which is not the case. In an irrotational flow the condition $\Delta = 0$ would mean $q^2 = 0$ which is meaningless.

IV. The Flow Solutions Lost in the Molenbroek-Chaplygin Transformation

The procedure applied by Tollmien (1941) will be adjusted in the present case to two types of flow. The vanishing determinant $\partial(q, \theta)/\partial(x, y)$ is a criterion for the lost solutions.

In order to evaluate these "lost solutions" in a viscous flow the continuity equation and the vorticity equation ($v_x - u_y = \omega$) are used. The partial differential co-efficients, e_x and e_y are calculated from equation (23) with the use of the equality: $\frac{1}{2}Tc_p^{-1} = (\gamma R)^{-1}c^2$. From the identical vanishing of the functional determinant, θ will be a function of q alone (both functions are "functionally" related); that is, $\theta = \theta(q)$. Using the relations:

$$u_x = q_x \cos \theta - q \sin \theta (d\theta/dq) q_x, \quad dh_{01}/dq = h_{01q} + h_{01\theta} (d\theta/dq), \text{ etc.},$$

leads to the following two equations:

$$q_x \{[(c^2 - q^2) + Hq] \cos \theta - c^2 q \sin \theta (d\theta/dq)\} + q_y \{[(c^2 - q^2) + Hq] \sin \theta + c^2 q \cos \theta (d\theta/dq)\} = q_x I_1 + q_y I_2 = 0, \quad (45a)$$

$$q_x [\sin \theta + q \cos \theta (d\theta/dq)] - q_y [\cos \theta - q \sin \theta (d\theta/dq)] = \omega, \quad (45b)$$

$$H = dh_{01}/dq - c^2 (\gamma R)^{-1} (dS/dq).$$

Consider equations (45) as a linear non-homogeneous set in q_x and q_y . The determinant is equal to $\Delta_1 = (q^2 - c^2) - c^2 q^2 (d\theta/dq)^2 - H$. The application of Cramer's rule gives:

$$q_x = -(\omega/\Delta_1) I_2, \quad q_y = (\omega/\Delta_1) I_1. \quad (46)$$

Take lines of constant velocity: $q = \text{const}$, or $dq = q_x dx + q_y dy = 0$. Hence one obtains:

$$dy/dx = -(q_x/q_y) = I_2/I_1. \quad (47)$$

Equation (47) together with the value of $d\theta/dq$ calculated from equation (38) defines the lines of constant velocity. Since θ depends only on q the lines of constant velocity are

straight lines. Along these lines ψ is variable, whereas q and θ are constant, hence ψ cannot be considered as a function of q and θ , which is a condition for the Molenbroek-Chaplygin transformation. These lines, obtained as "lost" solutions, generally do not correspond to Mach lines which may easily be proved. Equation (46) represents the trivial solution of the set (45). In order to obtain the indeterminate values of q_x and q_y ($q_x = q_y = 0/0$) corresponding to non-trivial solutions of (45) one must have $\Delta_1 = I_1 = I_2 = 0$. The Mach lines may be defined by the relation

$$dy/dx = \tan(\theta \pm \epsilon), \quad (48)$$

but the conditions $I_1 = I_2 = 0$ give the result: $\tan \theta = -\cot \theta = \tan(90^\circ + \theta)$, which cannot be fulfilled by any real angle including zero. Hence there does not exist any line of $q = \text{const.}$, which coincides with a Mach line for $\omega \neq 0$. In one and only one case the lines of constant velocity correspond to Mach lines, namely, if and only if $\omega = H = 0$. In this case the set (45) has as the trivial solution $q_x = q_y = 0$ (i.e., q is identically constant) and the non-trivial solutions may be obtained from the condition $\Delta_1 = 0$ which inserted into equation (47) gives equation (48) as the result without any restrictions superimposed upon the value of the angle θ . In the case of an inviscid, rotational flow the use of the "modified" continuity equation leads to equations similar to those derived above with $H = 0$. The condition $\Delta_1 = 0$ inserted into (47) leads directly to equation (48). The conditions $I_1 = I_2 = 0$ again lead to the conclusion derived above (see VIII).

V. Limiting Lines

V.1. The Condition for a Limiting Line. The single-valued functional relationship of the transformation is not assured when the functional determinant $\partial(x, y)/\partial(q, \theta)$ vanishes. Calculate this determinant from equation (35):

$$\Delta_2 = \alpha^2 [d_1 \cos \theta - c_1 \sin \theta] [a_2 \psi_q^2 - (a_1 + b_2) \psi_q \psi_\theta + b_1 \psi_\theta^2]. \quad (49)$$

The second factor on the right hand side of equation (49) is equal to $q^{-2} \Delta$ and consequently, cannot be equal to zero (see III.3). The only possibility is that the third factor on the right-hand side is equal to zero. Introducing as the independent variables ξ and η transforms this factor, denoted by Z , into:

$$Z = \bar{\alpha} \psi_\xi^2 + \bar{\beta} \psi_\xi \psi_\eta + \bar{\gamma} \psi_\eta^2 = 0. \quad (50)$$

Equation $Z = 0$ is satisfied when:

(a) $\psi_\xi = \psi_\eta = 0$, or $\psi_q = \psi_\theta = 0$. In this case the total differential $d\psi = \psi_q dq + \psi_\theta d\theta$ is identically equal to zero. On the other hand, this total differential is given by equation (29b). Since the ratio of dy to dx is arbitrary one must have $u = v = 0$, in order that $d\psi = 0$. This may happen in a singular point, that is, in a stagnation point, consequently in the subsonic region.

(b) Equation $Z = 0$ is fulfilled for $a_2 \equiv (a_1 + b_2) \equiv b_1 \equiv 0$. This case leads to the condition: $\bar{\alpha} \equiv \bar{\beta} \equiv \bar{\gamma} \equiv 0$, which is case (c).

(c) The case $\bar{\alpha} \equiv \bar{\beta} \equiv \bar{\gamma} \equiv 0$ must be excluded since it is a contradiction to canonical forms, both hyperbolic and parabolic. Consequently it has been shown that the factor Z cannot vanish identically in a certain region with the surface area different from zero [16].

(d) The factor Z may vanish along a certain curve, the "limiting hodograph" (the corresponding curve in the physical plane is "limiting line.") In the neighborhood of a limiting line to a pair of x - y -values there correspond various q - θ -values and also various values of ψ . The integral surface $\psi(x, y)$ possesses on the limiting line a branch line. Thus the vanishing functional determinant $\partial(x, y)/\partial(q, \theta)$ leads to the equation of the "limiting hodograph" $Z = 0$, which will be called the "limiting equation." To obtain the real values of $Z = 0$, the discriminant must be non-positive. The remark in the section III.2 has shown that this restricts the existence of the limiting lines to the supersonic region only.

V.2. Rate of Pressure on the Limiting Line Along a streamline $\psi = \text{const}$; hence $\psi_q dq + \psi_\theta d\theta = 0$, or $d\theta = -(\psi_q/\psi_\theta) dq$. Substituting the last value into equations (35) gives the components dx and dy of a line element of a stream-line. Taking the value " q " as a parameter along a streamline, one obtains from the relation $\widehat{ds}^2 = \widehat{dx}^2 + \widehat{dy}^2$ the formula for the line element of a streamline:

$$dx = \alpha\psi_\theta^{-1}Z \cos \theta dq, \quad dy = \alpha\psi_\theta^{-1}Z \sin \theta dq, \quad ds = \alpha\psi_\theta^{-1}Z dq. \quad (51)$$

From equation (8a) using equation (5) one obtains the relation

$$\text{grad } \bar{p} = \mathbf{P} - \bar{q}[\mathbf{A} + \text{grad } (\tfrac{1}{2}V^2)].$$

Applying dimensionless quantities and keeping in mind that the vector \mathbf{A} has no component along a streamline, one arrives at the formula for the drop in pressure along a streamline;

$$p_s = \xi_1 - eqq_s, \quad \text{or} \quad p_s = \xi_1 - eq\psi_\theta(\alpha Z)^{-1}, \quad (52)$$

where ξ_1 denotes the sum of all the components of the stress tensor taken along a streamline. Let us discuss equation (52).

The quantities e and q are not equal to zero and are positive. But, of course, q may become very large on the limiting line. The value of α is finite, even if q is equal to zero, and tends to zero when q approaches infinity. Hence, assuming that ψ_θ is different from zero, it is obvious that the second term in equation (52) approaches $-\infty$ when $q_s > 0$ and when the limiting equation $Z = 0$ is satisfied. In the case of an inviscid rotational flow the term ξ_1 equals zero and one obtains $p_s \rightarrow -\infty$ on the limiting line (an analogy to a potential flow). In the case of a viscous flow the coefficient of viscosity is a function of temperature (the variation with pressure being neglected) and is finite. The components of the stress tensor, due to its structure, are functions of q_{ss} , q_s , q_{nn} , q_n and g_n and g_1 , g_2 , if a line element \widehat{dl} is given by the relation: $\widehat{dl}^2 = g_1^2 \widehat{ds}^2 + g_2^2 \widehat{dn}^2$ (g_1 and g_2 are positive), and if the velocity components are expressed in the general orthogonal curvilinear coordinate system s - n . Apart from the assumption made in

Section III.2. (that the second derivatives of the velocity components are functions of Φ, x, y), there is another relationship between q_{ss} and q_s quantities. Namely, represent the quantities q_{ss} by the expression $r^{-1}(1+q_s^2)^{3/2}$, where r denotes the radius of curvature. The quantity r is a function of the position (x, y) and without loss of generality one may assume that the value $|r|$ is a bounded quantity in a closed interval with zero as the minimum and a certain value, even very large, as the maximum. It is not probable that r may approach infinity in a neighborhood of the limiting line. Combining all the terms on the right side of equation (52) one obtains a general expression of the form (with $q_s > 0$):

$$\pm b_1(1+q_s^2)^{3/2} \pm b_2q_s + b_3q_{ss} + b_4q_{ss}, \quad (53)$$

where the variable coefficients b_i 's are functions of g_1, g_2, μ, T, r , etc., but not of q_s . In case the first and second terms have the same sign, the result for $Z = 0$ or $q_s \rightarrow \infty$ is a trivial one: $p_s \rightarrow \pm \infty$. In case the signs are opposite, an indeterminate form $\pm \infty \mp \infty$ results, provided the remaining terms are finite or zero. In order to find its value when $Z = 0$, consider the above expression (53) as a function of q_s (or Z) alone, multiply and divide it by $\mp b_1(1+q_s^2)^{3/2} \pm b_2q_s$, in order to obtain $(\pm \infty \mp \infty / \mp \infty \pm \infty)$, and apply l'Hospital rule. After several repetitions of this elementary procedure one gets the result that the limit is definitely equal to $\pm \infty$ with the sign equal to the sign of the first term in (53) (the first term in that expression tends to ∞ at a higher order than the second one). Hence the value $p_s \rightarrow \infty$ when the limiting equation is satisfied (for further discussion see Section VII).

The considerations given below follow those in [16, 1941]. Discuss the impossibility of the vanishing of ψ_s that is, the existence of a singular point ($p_s \approx 0/0$). As the condition for the existence of a singular point ($dy/dx = 0/0$) on a streamline one again obtains the limiting equation. If ψ_s assumes the value zero then with $\psi_t \neq 0$, a_2 must equal zero in order to satisfy the limiting equation. But this case must be excluded. With the assumption that $\psi_s = a_2 = 0$, eq. (33a) gives $\Phi_s = 0$ which would mean that $\Phi = \Phi(q)$ only. But if $\psi = \psi(q, \theta)$ then equation (33b) gives the result $\Phi = \Phi(q, \theta)$, which is a contradiction. The type of singular point where the streamline meets the limiting line can be determined when it is recalled that the direction of a streamline is given without any ambiguity by the independent variable θ in the hodograph plane. A streamline can either cross the limiting line as a smooth curve without any discontinuities in its first derivatives or may return in the same direction (the same tangent) after creating a cusp (saber point). In general this last case occurs, [16], because the equation $Z = 0$ has an ordinary zero (root) on the limiting line whereas ψ_s is not equal to zero at this point. That means that the equation $Z = 0$ changes its sign when passing through the limiting line (zero) and consequently dx and dy also change signs, i.e., the streamlines turn back from the limiting line. In order to show that the mentioned expression usually has an ordinary zero at the point under discussion, keep in mind that in general the expression on the left hand side of the equation $Z = 0$ is equal to zero on the limiting line, and take the derivative of this expression along a streamline

with respect to q , say, using the relation $d\theta = -(\psi_q/\psi_\theta)dq$. A more rigorous proof will be given at the end of Section V.3. Here the discussion will be restricted to a few remarks.

Neither the insertion of the hodograph equation nor that of the limiting equation will make the derivative vanish. This means that the equation $Z = 0$ changes its sign on the limiting line and the incoming flow turns back from the limiting line after creating a cusp. Thus the limiting line is the locus of the cusps of the streamlines.

V.3. Transformation of Mach Lines in the Hodograph Plane. Transform equation (87) onto the hodograph plane by substituting equations (35) into it. After all the necessary transformations one obtains.

$$c^2\{[(b_2 - e_1)\psi_q - b_1\psi_\theta]dq - [(a_1 + e_1)\psi_\theta - a_2\psi_q]d\theta\}^2 + (c^2 - q^2)f_1^2(\psi_q dq + \psi_\theta d\theta)^2 = 0, \quad (54a)$$

or

$$c^2\{[(b_2 - e_1)dq + a_2d\theta]\psi_q - [b_1dq + (a_1 + e_1)d\theta]\psi_\theta\}^2 + (c^2 - q^2)f_1^2(\psi_q dq + \psi_\theta d\theta)^2 = 0, \quad (54b)$$

$$e_1 = c_1 \cos \theta + d_1 \sin \theta, \quad f_1 = -q^{-1} \Delta.$$

Consider equations (54a) and (54b) as the quadratic equations in $dq/d\theta$ or $d\theta/dq$ and ψ_q or ψ_θ respectively. The discriminants of both those equations are:

$$\begin{aligned} -4c^2 d'_8 (d_6 \psi_\theta + d_7 \psi_q)^2 & \quad \text{and} \quad -4c^2 d'_8 (e_6 d\theta + e_7 dq)^2, \\ d_6 &= (b_2 - e_1)\psi_q - b_1\psi_\theta, & e_6 &= (b_2 - e_1)dq + a_2d\theta, \\ d_7 &= (a_1 + e_1)\psi_\theta - a_2\psi_q, & e_7 &= (a_1 + e_1)d\theta + b_1dq, \\ d_8 &= (c^2 - q^2)f_1^2 = -d'_8. \end{aligned}$$

Hence in the supersonic region both discriminants are negative. Calculate from equation (54a) the values $dq/d\theta$ and $d\theta/dq$. Comparing the values $1/(dq/d\theta)$ and $d\theta/dq$ one obtains the equation:

$$(cd'_8 K_1 \pm d_8^{1/2} K_2) K_1 = 0, \quad (54c)$$

$$K_1 = d_6 \psi_\theta + d_7 \psi_q, \quad K_2 = c^2 d_8 d_1 + d'_8 \psi_q \psi_\theta.$$

If $K_1 = 0$, then $Z = 0$. If the expression inside the bracket (54c) vanishes, then $K_1 = K_2 = 0$, as a result of the double sign. Exclude the case $c^2 = q^2$ or $d_8 = 0$ for the reason explained above. Also exclude the case $c^2 = 0$ since it corresponds to a vacuum. The factor f_1 cannot vanish. After the necessary transformations the condition $K_2 = 0$ takes the form:

$$\begin{aligned} \delta_2 \psi_q^2 + \delta \psi_q \psi_\theta + \delta_1 \psi_\theta^2 &= 0, \\ \delta &= [(b_2 - e_1)(a_1 + e_1) + a_2 b_1 + d'_8], \\ \delta_1 &= -b_1(a_1 + e_1), \quad \delta_2 = -(b_2 - e_1)a_2. \end{aligned} \quad (54d)$$

Since equation $Z = 0$ must be satisfied, equation (54d) cannot hold. Hence $K_2 \neq 0$ and the only result is $K_1 = 0$. The same technique applied to (54b) gives equation (38) as the result. Identically the same result is obtained, if one assumes that in order that each of equations (54a) and (54b) be satisfied, both terms in each equation should equal zero. If $d_8 \neq 0$, this results in a set of two linear homogeneous equations in ψ_q and ψ_θ .

or dq and $d\theta$. In order that each of these two sets has a non-trivial solution, i.e., in order that ψ_q and ψ_θ or dq and $d\theta$ are not identically equal to zero, the determinants must be zero; namely,

$$\begin{vmatrix} (b_2 - e_1)dq + a_2d\theta & -b_1dq - (a_1 + e_1)d\theta \\ dq & d\theta \end{vmatrix} = 0, \text{ etc.}$$

The final results are again equations $Z = 0$ and (38). Thus the transformation of the equation of the characteristic curves from the physical plane into the hodograph plane gave two possible cases:

- (a) the characteristic hodograph (38),
- (b) the limiting equation $Z = 0$ and limiting lines.

Hence one is confronted with the existence of two types of Mach lines (a) and (b).

In order to investigate the behavior of a family of Mach waves on the limiting line, study the behavior of a line element of a Mach wave. For a family of Mach waves in the hodograph plane one may calculate the value of the derivative $d\theta/dq$ from equation (38):

$$d\theta = A_3 dq, \quad A_3 = \{-(a_1 + b_2) \pm [(a_1 + b_2)^2 - 4a_2b_1]^{1/2}\}(2a_2)^{-1}.$$

There are two signs before the root depending on which family is used. Inserting that value and equations (33) into equations (35) gives:

$$dx = \alpha [c_1(\psi_q + A_3\psi_\theta) + e \cos \theta (\Phi_q + A_3\Phi_\theta)] dq. \quad (55)$$

In order to obtain dy , substitute into equation (55) d_1 and $\sin \theta$ instead of c_1 and $\cos \theta$, respectively.

Calculating the values Φ_q and Φ_θ from equations (38) and inserting them into equation $Z = 0$ one gets the result (similar result may be obtained for $\omega = \mathbf{P} = 0$) (exclude the case $c_1/d_1 = \cot \theta$):

$$a_2\Phi_q^2 - (a_1 + b_2)\Phi_q\Phi_\theta + b_1\Phi_\theta^2 = 0. \quad (56)$$

For a singular point on the Mach line ($dy/dx = 0/0$) dy and dx must vanish simultaneously. Consequently there results:

$$\psi_q + A_3\psi_\theta = \Phi_q + A_3\Phi_\theta = 0. \quad (57)$$

As may easily be verified, condition (57) is fulfilled for the limiting equation and for equation (56). Hence the Mach lines have singular points on the limiting line. The same reasoning as that used above for the streamlines, shows readily that the Mach lines possess cusps (saber points) on the limiting line. Similarly as in the case of an irrotational flow in order to prove that a limiting line is an envelope for another family of Mach waves, it is sufficient to show that the limiting hodograph $Z = 0$ is not a characteristic hodograph (38). If this were the case then the two equations (38) and $Z = 0$ would have to be simultaneously satisfied, which condition would imply that $\psi_q/\psi_\theta = -d\theta/dq$. But this is the relation for a directional tangent of streamline, since the coincidence and the accordance of directions are transferable from the hodograph plane into the physical plane, whereas an opposite transformation in the neighborhood of a limiting line is not valid. This would mean that the limiting line coincides with a

streamline, but a limiting line as a Mach line can coincide with a streamline only in a vacuum (see Section VI).

The proof that a flow cannot proceed beyond the limiting line is almost identical with that given by Tollmien for an irrotational flow. (See Section VI).

VI. Extension of Tollmien's proofs

Tollmien (1941) proved rigorously that the derivative of Z is not equal to zero. Consider $F(q, \theta) = Z = 0$. From the relation $F(q, \theta) = 0$, it follows that

$$d\theta/dq = -F_q/F_\theta. \quad (a)$$

Compute the partial derivatives F_q and F_θ from the expression $F(q, \theta) = Z$. Combine (a) with the expression $\psi_q/\psi_\theta = -d\theta/dq$, and obtain an inequality

$$F_q\psi_\theta - F_\theta\psi_q \neq 0. \quad (b)$$

The left hand side of (b) is exactly equal to the derivative of Z . The note of Tollmien (1941) is also of a significance. Equations (36) and (18) in Tollmien's paper are exactly satisfied in the case discussed in the present paper. Instead of the value of the determinant below equation (38) in his paper one obtains the value

$$(c_1 \sin \theta - d_1 \cos \theta) \left[a_2 \left(\frac{d\theta}{d\sigma} \right)^2 + (a_1 + b_2) \left(\frac{dq}{d\sigma} \right) \left(\frac{d\theta}{d\sigma} \right) + b_1 \left(\frac{dq}{d\sigma} \right)^2 \right],$$

in the present case. This is again the expression for the characteristic hodograph (38).

VII. Hypothesis of Shock in Viscous Gases

Write equation (52) in the form

$$(eq)^{-1} p_s = R_e^{-1} \xi_2 - q_s, \quad (58)$$

where R_e denotes Reynolds number $req\mu^{-1}$ and $\xi_2 = r\xi_1$, and consider an accelerated motion, i.e., $p_s < 0$, $q_s > 0$. From equation (58) one concludes that in an inviscid flow $p_s \rightarrow (-\infty)$ on the limiting line, whereas in a viscous flow $p_s \rightarrow +\infty$ due to the fact that the term $R_e^{-1}\xi_2$ is predominant on the limiting line. This behavior suggests the following hypothesis of shock in viscous gases: Start with a uniform flow at which $q_s = 0$. Due to the superimposed boundary conditions (for example, a convergent-divergent nozzle), the value of q_s increases and is positive and that of p_s is negative. For "relatively small" values of q_s the term $R_e^{-1}\xi_2$ which is proportional to higher powers of q_s , is a small one. Thus one has conditions rather similar to an inviscid flow ($-\infty$ on the limiting line). With an increase in q_s , however, the significance of the term $R_e^{-1}\xi_2$ increases, the value of p_s passes through a minimum value and the curve $p_s = p_s(q_s)$ turns upward. That region corresponds to a viscous flow (with $+\infty$ on the limiting line). The point $p_s = 0$ is reached but due to the great mass inertia of the incoming flow and the compressibility effects (which do not occur in an incompressible flow) there appears a ram phenomenon, i.e., p_s takes positive values, density increases, q_s decreases, and may be negative, until $p_s = p_s(q_s)$ reaches a maximum value. From that point q_s

increases and is positive, p_s decreases, reaches the value $p_s = 0$ and drops below the q_s -axis, and the procedure repeats. The significance of R_s and q_n may also be investigated. The latter quantity is a very significant one in the boundary layer and if its value is a predominant one, a shock may appear close to or in the boundary layer as tests show [1, 10]. The change in sign of q_s (which occurs at $p_s > 0$), or of e_s may be taken as the beginning of the shock, the end of it being dictated by the thermodynamic phenomena inside the shock, here not outlined (an analogy to a hysteresis loop).

VIII. Final Remarks

It is obvious that the procedure and proof presented above may be applied to any generalized velocity potential, even if the superimposed vector field is proportional to m -th order partial derivatives of the velocity components provided the considered functions (velocity, temperature, density, etc.) satisfy the original partial differential equations together with the boundary conditions.

In each case, instead of the considerations, presented in Section III.2, one may briefly assume that the right-hand sides of equations (2d) and (27) are certain (unknown) regular functions of q , x , and y , which is certainly true. The knowledge of the form of these functions is not necessary to extend Tollmien's approach, which was done up to the very end of his considerations; but considerations in Section VII show clearly that a limiting line cannot occur in a viscous flow due to the change of sign of p_s . That change of sign does not occur in an inviscid flow. Thus, only in a viscous flow, one is able to show why a limiting line does not appear, and, namely, that the non-occurrence of a limiting line is due to, and only to, the terms containing viscous stresses (eq. 52). Strictly speaking, equation (52) is a Navier-Stokes equation taken along a streamline. This justifies the use of symbols in the entire procedure. The terms expressing the shearing stresses may be expanded at the very end. In an inviscid fluid we are able to show that a limiting line does not occur, but we are unable to prove why it does not appear. In a viscous flow, the superposed perturbation vector field in the neighborhood of the sonic speed is a predominant one and changes basically the flow pattern of a potential flow, i.e., a limiting line does not occur.

Another interesting problem may be to investigate the influence of the condition of the vanishing Jacobian upon the singularities of the original third-or-higher order partial differential equation. Finally, the continuation of Wasow's idea [19], i.e., the investigation of the relation between the asymptotic solutions of the original and of the reduced order partial differential equations offers a vast field for mathematicians.

An integral surface of the differential equation of a potential supersonic flow possesses branchlines (characteristics). As shown above, the corresponding surface in case of a viscous supersonic flow possesses analogous branchlines due to second-order terms, and moreover, has singularities due to third-order terms. A study of the mutual relationship of those second and third order singularities may be interesting. Finally, practically an important problem is to find an approximate solution of the equations of

viscous flow and to define the finite "front-line" of the shock (where the word, "shock", denotes a steep transition in p , T , etc.) as the locus of the points in which $p_s = 0$ (eq. 52). The influence of the acceleration pattern around a body, say, is obvious. The type of the shock (detached or attached) will come out by itself.

Note added in proof correction (9. 9. 50). The meaning of the limiting lines depends upon the existence of intrinsic hodograph equations for r and y as functions of q and θ . These equations are given by (35). Since the coefficients in (35) may be complicated functions of ψ , q , θ , ω , etc., equations (35) may be very complicated and of higher order. But if a solution exists then those coefficients are certain functions (unknown at the beginning if the solution is unknown) of q and θ only. Similarly (36) is fundamentally a non-linear equation with coefficients being functions of third partial derivative of ψ (viscous flow). But again, if a solution exists they may be assumed to be functions of q and θ only (reduced order equation). Although we are not able to solve (36) we may discuss the characteristics of the reduced order equation. Finally, let us explain briefly (47) and (48) with $\Delta_1 = H = 0$. In this case $(q\theta_q)^2 = (\bar{q}^2 - c^2)\bar{c}^{2-2} = \cot^2 \epsilon$.

Inserting this quantity into the formula

$$dy/dx = -q_x/q_y = [(c^2 - q^2) \sin \theta + c^2 q \theta_q \cos \theta] [(c^2 - q^2) \cos \theta - c^2 q \theta_q \sin \theta]^{-1},$$

we easily obtain the result:

$$(\cot \epsilon \sin \theta \pm \cos \theta)(\cot \epsilon \cos \theta \mp \sin \theta)^{-1} = (\cot \epsilon \tan \theta \pm 1)(\cot \epsilon \mp \tan \theta)^{-1},$$

or

$$(\tan \theta \pm \tan \epsilon)(1 \mp \tan \theta \tan \epsilon)^{-1} = \tan (\theta \pm \epsilon).$$

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ON SOME RELATIONS INVOLVING LAGUERRE POLYNOMIAL $L_n(z)$

BY

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Introduction

This short paper, as its name indicates, deals with a series of relations existing among the set of Laguerre polynomials $\{L_n(z)\}$. For convenience, the paper has been divided into two sections; in Section I, we have shown that starting from some well-known relations between the Whittaker's Confluent Hypergeometric function $W_{k,m}(z)$, it is possible to derive some interesting relations existing amongst the set of Laguerre polynomials $\{L_n(z)\}$, *one amongst them being that it is possible to express $L_n(u+v)$ in the form of an infinite series involving $L_r(u)$ or $L_r(v)$.*

In Section II, an attempt has been made to express Laguerre polynomial in terms of Sonine's polynomial in the form discussed by Basu (1943) and *vice versa*. It may be noted that the definition as adopted by Basu, is slightly *different* from that made in *Whittaker and Watson's Course of Modern Analysis* (1940) and represented as $T_m^n(z)$.

Finally, a form of *Addition Theorem* for the function $L_n(z)$ has been obtained.

It is believed that a considerable portion of the results incorporated in this paper are original although in some cases, particularly in Section II some influence of the methods adopted by Basu (1943) may be noticed.

Section I

1. It is well-known (Hobson 1931, p. 88) that, for $|h| < 1$

$$\frac{e^{-sh/(1-h)}}{1-h} = \sum_{n=0}^{\infty} \frac{h^n}{n!} L_n(z) \quad (1)$$

where $L_n(z)$ is the Laguerre polynomial defined by

$$L_n(z) = e^z \frac{d^n}{dz^n} [e^{-z} z^n].$$

Further, with $|h| < 1$, we have also the expansion

$$\frac{e^{-sh/(1-h)}}{1-h} = z^{-\frac{1}{2}} e^{\frac{1}{2}z} \sum_{n=0}^{\infty} \frac{(-h)^n}{n!} W_{n+\frac{1}{2},0}(z) \quad (2)$$

[Goldstein, 1932, p. 112], where $W_{k,m}$ is Whittaker's Confluent Hypergeometric function.

A comparison of the coefficients of h^n in (1) and (2) leads immediately to

$$W_{n+\frac{1}{2},0}(z) = (-1)^n z^{\frac{1}{2}} e^{-\frac{1}{2}z} L_n(z)^*. \quad (3)$$

Starting next with the relation [Goldstein, 1932, p. 112], viz.,

$$\alpha^k e^{-\frac{1}{2}\alpha z} W_{k,m}(\alpha z) = e^{\frac{1}{2}z} \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{1}{\alpha} - 1 \right)^r W_{k+r,m}(z) \quad (4)$$

where α is non-negative, we put $k = n + \frac{1}{2}$ and $m = 0$. Then making use of the relation (8), we obtain from (4) after some easy steps, the relation :

$$\alpha^{n+1} L_n(\alpha z) = e^{\alpha(z-1)} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{1}{\alpha} - 1 \right)^r L_{n+r}(z). \quad (5)$$

Replacing z by v and α by $1+u/v$, where u and v are so chosen that α may remain positive, we immediately derive, after simplification, the result :

$$L_n(u+v) = \frac{e^u v^{n+1}}{u+v} \sum_{r=0}^{\infty} \frac{u^r L_{n+r}(v)}{r! (u+v)^{n+r}}. \quad (6)$$

Interchanging u and v , we can also write the result (6) as

$$L_n(u+v) = \frac{e^v u^{n+1}}{u+v} \sum_{r=0}^{\infty} \frac{v^r L_{n+r}(u)}{r! (u+v)^{n+r}} \quad (7)$$

(6) or (7) thus expresses $L_n(u+v)$ in the form of an infinite series involving $L_n(v)$ or $L_n(u)$.

Having thus obtained a formula for expressing $L_n(u+v)$ as an infinite series, we proceed in the following article, to deduce some of the simple relations involving $L_n(z)$.

2. Case (i). In the relation for $W_{k,m}$ function [Whittaker & Watson, 1940 p. 352 Ex 3 (ii)], viz.

$$z W'_{k,m}(z) = (k - \frac{1}{2}z) W_{k,m}(z) - \{m^2 - k - \frac{1}{2}\} W_{k-1,m}(z), \quad (8)$$

put $k = n + \frac{1}{2}$ and $m = 0$ so as to derive

$$z \frac{d}{dz} [W_{n+\frac{1}{2},0}(z)] = (n + \frac{1}{2} - \frac{1}{2}z) W_{n+\frac{1}{2},0}(z) + n^2 W_{n-\frac{1}{2},0}(z). \quad (9)$$

The relation (9) taken in conjunction with (8) leads after simplification, to the known result†

$$z L'_n(z) = n L_n(z) - n^2 L_{n-1}(z). \quad (10)$$

Case (ii). Again combining together the recurrent relation [Whittaker & Watson, 1940, p. 352, Ex. 3(1)], viz.

$$W_{k,m}(z) = z^{\frac{1}{2}} W_{k-\frac{1}{2},m-\frac{1}{2}}(z) + (\frac{1}{2} - k + m) W_{k-1,m}(z) \quad (11)$$

in which k is replaced by $n + \frac{1}{2}$ and m by 0, and the relation (8) of Art. 1, we easily obtain the following result :

$$L_n(z) - n L_{n-1}(z) = (-1)^n z^{\frac{1}{2}} W_{n,-\frac{1}{2}}(z) \quad (12)$$

* This result becomes known if we simply take into consideration the fact that the function $\phi_n(z)$ defined by Lagrange and Abel is simply $\phi_n(z) = L_n(z)/n!$ (Whittaker and Watson, 1940 p. 353 Ex. 9).

† The relation (10) although worked out in Courant and Hilbert (1931) is added here simply to illustrate the fact that the same may be obtained as a corollary to the general $W_{k,m}$ -function.

Since $W_{k,n}(z) = W_{k,-n}(z)$ [Whittaker and Watson, 1940, p. 370], the relation (12) can also be written as

$$L_n(z) - nL_{n-1}(z) = (-1)^n e^{\frac{1}{2}z} W_{n, \frac{1}{2}}(z). \quad (13)$$

On substitution of the value of $W_{n, \frac{1}{2}}(z)$ given in terms of Bateman's k -function, *vis.*

$$\Gamma(1+n)k_{2n}(\tfrac{1}{2}z) = W_{n, \frac{1}{2}}(z)$$

(N. A. Shastri, 1938-39) in the relation (13) we immediately derive

$$L_n(z) - nL_{n-1}(z) = (-1)^n e^{z/2} \Gamma(1+n) k_{2n}(\tfrac{1}{2}z)$$

a result connecting Laguerre polynomials with Bateman's k -function.

In the following section we propose to devote ourselves to finding out a method for expressing the Laguerre polynomial in terms of Sonine's polynomial [Sonine, 1880] and *vice versa* and also an Addition Theorem for the function $L_n(z)$.

Section II

3. Sonine's polynomial $S_\lambda^n(z)$ being defined (Bose, 1943, p. 21) as

$$S_\lambda^n(z) = 1 - \frac{\binom{n}{1}}{\lambda_1} z + \frac{\binom{n}{2}}{\lambda_2} z^2 - \frac{\binom{n}{3}}{\lambda_3} z^3 + \dots + (-1)^n \frac{\binom{n}{n}}{\lambda_n} z^n,$$

where

$$\binom{n}{r} = \frac{n(n-1) \cdots (n-r+1)}{r!}, \quad \lambda_r = \lambda(\lambda+1) \cdots (\lambda+r-1)$$

and λ is a rational positive quantity. We know (Bose, 1943, p. 21) that

$$\frac{e^{-zt/(1-t)}}{(1-t)^\lambda} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} S_\lambda^n(z), \quad (|t| < 1). \quad (14)$$

A glance at the results (1) and (14) at once reveals that

$$\Gamma(1+n) S_1^n(z) = L_n(z), \quad (15)$$

which clearly shows that relations involving Laguerre polynomials may be obtained as particular cases from those involving Sonine's polynomials.

In what follows, the Laguerre polynomial will be expressed as a series involving Sonine's polynomial and *vice versa*.

4. Since

$$\frac{e^{-sh/(1-h)}}{1-h} = \frac{e^{-sh/(1-h)}}{(1-h)^\lambda} \cdot \frac{1}{(1-h)^{1-\lambda}},$$

Therefore, on the strength of the relations (1) and (14), we obtain

$$\sum_{n=0}^{\infty} \frac{h^n}{n!} L_n(z) = \sum_{m=0}^{\infty} \frac{h^m \Gamma(\lambda+m)}{m! \Gamma(\lambda)} S_\lambda^m(z) \sum_{r=0}^{\infty} \frac{h^r}{r!} \frac{\Gamma(1-\lambda+r)}{\Gamma(1-\lambda)},$$

where λ is assumed to be less than unity. Whence, on equating the coefficients of h^n from both the sides, we readily obtain

$$\frac{L_n(z)}{n!} = \sum_{v=0}^n \frac{\Gamma(\lambda+n-v)\Gamma(1-\lambda+v)}{\Gamma(\lambda)\Gamma(1-\lambda)v!(n-v)!} S_{\lambda}^{n-v}(z) \quad (16)$$

which is the relation expressing a Laguerre polynomial in terms of Sonine's polynomials.

Again, to express Sonine's polynomial as a series of Laguerre polynomials we observe that

$$\frac{e^{-sh/(1-h)}}{(1-h)^s} = \frac{e^{-sh/(1-h)}}{1-h} \cdot \frac{1}{(1-h)^{s-1}}$$

so that, assuming $\mu > 1$ and appealing to the relations (1) and (14) we readily derive

$$\sum_{n=0}^{\infty} \frac{h^n}{n!} \frac{\Gamma(\mu+n)}{\Gamma(\mu)} S_{\mu}^n(z) = \sum_{m=0}^{\infty} \frac{h^m}{m!} L_m(z) \sum_{r=0}^{\infty} \frac{h^r}{r!} \frac{\Gamma(\mu-1+r)}{\Gamma(\mu-1)}$$

On equating the coefficients of h^n from both sides, we have, clearly,

$$\frac{S_{\mu}^n(z)}{n!} = \sum_{r=0}^n \frac{\Gamma(\mu)\Gamma(\mu-1+r)}{\Gamma(\mu-1)\Gamma(\mu+n)r!(n-r)!} L_{n-r}(z) \quad (17)$$

which expresses $S_{\mu}^n(z)$ in terms of $L_n(z)$ in the form of a finite series.

We shall now conclude this paper by adding the following article on the "*Addition Theorem for the Laguerre polynomial*" which is derived from a very simple consideration viz. the generating function of the Laguerre polynomials.

5. From the relations :

$$\frac{e^{-uh/(1-h)}}{1-h} = \sum_{m=0}^{\infty} \frac{h^m}{m!} L_m(u). \quad (18)$$

and

$$\frac{e^{-vh/(1-h)}}{1-h} = \sum_{n=0}^{\infty} \frac{h^n}{n!} L_n(v) \quad (19)$$

we have, by multiplication,

$$\frac{e^{-(u+v)h/(1-h)}}{(1-h)^2} = \sum_{m=0}^{\infty} \frac{h^m}{m!} L_m(u) \sum_{n=0}^{\infty} \frac{h^n}{n!} L_n(v) \quad (20)$$

which is the same thing as

$$\sum_{r=0}^{\infty} \frac{h^r}{r!} L_r(u+v) = (1-h) \sum_{m=0}^{\infty} \frac{h^m}{m!} L_m(u) \sum_{n=0}^{\infty} \frac{h^n}{n!} L_n(v). \quad (21)$$

Equating the coefficients of h^r from both sides of (21) the desired addition theorem for the Laguerre polynomial is seen to be

$$\frac{L_r(u+v)}{r!} = \sum_{v=0}^r \frac{L_r(v)L_{r-v}(u)}{v!(r-v)!} - \sum_{v=0}^{r-1} \frac{L_v(v)L_{r-v-1}(u)}{v!(r-v-1)!}.$$

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ON AN ALGEBRAIC SYSTEM GENERATED BY A SINGLE ELEMENT AND ITS APPLICATION IN RIEMANNIAN GEOMETRY—II

By

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1. This is a continuation of a previous paper (Sen, 1950) in which an attempt was made to construct an algebraic system and to study its properties. It was shown that the application in Riemannian geometry of some of the properties of the above system yielded some interesting results including the identification of the Christoffel symbol. The present paper aims at further information on the use and application of the system. It is shown that the system may be used to construct newer elements possessing, broadly speaking, two kinds of properties which may be applied to identify curvature tensors in Riemannian geometry. It seems desirable to begin this paper with a brief summary of those main features of the system, obtained in the paper referred to, which are considered necessary for the purpose of development of the present paper.

I. General nature of the system.

Let t be an abstract element and S be a system of elements generated by t in the following manner and having the following properties :

(1) Corresponding to every element a of S , there exists in S two elements called the *associate* and the *conjugate* of a and denoted by a^* and a' respectively. The associate and the conjugate shall be governed by the property that the associate of the associate as well as the conjugate of the conjugate of a is a :

$$a^{**} = a'' = a. \quad (1.1)$$

(2) A *commutative composition* is defined in S whereby every pair of elements a, b of S is composed to form an element $a \circ b$ of S , and the composition is such that for every element a (†)

$$a \circ a = a \quad (1.2)$$

(3) The associate and the conjugate of $a \circ b$ shall satisfy

$$(a \circ b)^* = a^* \circ b^*, \quad (a \circ b)' = a' \circ b'. \quad (1.3)$$

An element of S which is equal to its associate is called a *self-associate* element. Similarly for a *self-conjugate* element. Evidently, the elements

$$a \circ a^* \quad \text{and} \quad a \circ a' \quad (1.4)$$

are self-associate and self-conjugate respectively.

† This idempotent condition is introduced here for the sake of simplicity and unification.

II. Special conditions and properties

Let $a_1 = a$, $a_2 = a^*$, $a_3 = a^{*!}$, $a_4 = a^{*!*$, ...

(1) If the following sequence generated by the element a of S

$$a_1, a_2, a_3, a_4, \dots \quad (1.5)$$

is supposed to be finite, it is a *cyclic* sequence with an even number, say p , of terms.

Assume that the sequence (1.5) is finite with $p > 4$ and suppose that the terms of the sequence are distinct.

(2) The number of distinct terms of the following sequence of the type (1.5) generated by an element $a_i \circ a_j$ of S

$$a_i \circ a_j, (a_i \circ a_j)^*, (a_i \circ a_j)^{*!}, (a_i \circ a_j)^{*!/}, \dots$$

cannot exceed p and the nature of the sequence depends, among other things, on whether $p/2$ is even or odd.

(8) Assume that the system S contains just one element which has the property of being *both self-associate and self-conjugate*. Then, under certain circumstances, this unique element is

$$a_i \circ a_{+p/2}, \quad i = 1, 2, \dots, \quad (1.6)$$

provided that $p/2$ is even.

2. Let us suppose that all the conditions, assumptions and properties regarding the system S as stated in the last article are satisfied. Also we may state, as is generally the case with any composition, that if $a \circ b = a \circ c$, then $b = c$.

Now suppose that corresponding to every element a of S , there exists a new kind of element $f(a)$, defined as a function of a . The function is a 1-1 correspondence, $a \leftrightarrow f(a)$. Moreover, suppose that $f(a)$ represents a *permutation*, say the *identity*, of a group of permutations of three 'objects', s, i, j . Let this representation be expressed in writing by

$$f(a) = f_{sij}(a). \quad (2.1)$$

Then the element $f(a)$ gives rise to other elements corresponding to different permutations of the group. It is supposed that none of the new elements thus introduced belongs to S , but that the composition which is defined in S is applicable to these new elements also. These elements are assumed to have the following two properties:

If pqq is a given permutation of sij , chosen arbitrarily, then, when a runs over the elements of S ,

$$f_{pqq}(a) \circ f_{pqq}(a^*) = \varphi_{pqq} \quad (2.2)$$

remains *invariant* for the given permutation, and

$$f_{pqq}(a) = f_{pqq}(a'). \quad (2.3)$$

Obviously the invariant φ_{pqq} is symmetric in the indices p, q . Various formulae can be deduced from (2.2) and (2.3). For example,

$$\varphi_{pqq} = f_{pqq}(a) \circ f_{pqq}(a^*) = f_{pqq}(a^*) \circ f_{pqq}(a) = f_{pqq}(a') \circ f_{pqq}(a'^*) = f_{pqq}(a'^*) \circ f_{pqq}(a').$$

Writing a for a' we obtain

Similarly, since
$$f_{pqg}(a) \circ f_{gqp}(a'^*) = f_{pqg}(a'^*) \circ f_{gqp}(a) = \varphi_{gqp}. \quad (2.4)$$

we have

$$\begin{aligned} \varphi_{gqp} &= f_{gqp}(a) \circ f_{gqp}(a^*) = f_{gqp}(a^*) \circ f_{gqp}(a), \\ f_{gqp}(a) \circ f_{gqp}(a'^*) &= f_{gqp}(a'^*) \circ f_{gqp}(a) = \varphi_{gqp}. \end{aligned} \quad (2.5)$$

As special cases of (2.2) and (2.3), we have

$$f_{pqg}(a \circ a^*) \circ f_{gqp}(a \circ a^*) = \varphi_{gqp}, \quad f_{pqg}(a \circ a') = f_{gqp}(a \circ a') \quad (2.6)$$

If u is the unique element of S which is both self-associate and self-conjugate, we obtain from (2.4) and (2.5)

$$(f_{gqp}(u) \circ f_{pqg}(u)) \circ (f_{gqp}(u) \circ f_{pqg}(u)) = \varphi_{gqp} \circ \varphi_{gqp}. \quad (2.7)$$

3. We now introduce newer elements. Corresponding to every element a of S , let there exist an element $F(a)$, defined as another function of a , $a \mapsto F(a)$. Further suppose that $F(a)$ represents a permutation, say the identity, of a group of permutations of four 'objects', s, i, j, k , and let this representation be expressed by

$$F(a) = F_{sijk}(a). \quad (3.1)$$

Then $F(a)$ gives rise to other elements corresponding to different permutations. Every element thus created is supposed to be different from the elements introduced in the last two articles, but the composition which holds for the elements of S is supposed to hold for these newer elements also. These elements are assumed to have the following properties:

There is a 1-1 correspondence, $f(a) \leftrightarrow F(a)$, expressed by

$$f_{sij}(a) \leftrightarrow F_{sijk}(a). \quad (3.2)$$

If $pqgh$ is an arbitrary permutation of $sijk$, then, when a runs over the elements of S ,

$$F_{pqgh}(a) \circ F_{pqgh}(a) = O \quad (3.3)$$

remains *invariant independent of the permutation*. And, with reference to this invariant element O , there are also the following two properties:

$$F_{pqgh}(a) \circ F_{pqgh}(a^*) = O, \quad (3.4)$$

$$F_{pqgh}(a \circ a') \circ F_{pqgh}(a \circ a') = F_{pqgh}(a \circ a') \circ O. \quad (3.5)$$

Various formulae can be derived from the above properties. For example, it follows from (3.3) and (3.4) that

$$F_{pqgh}(a) = F_{pqgh}(a^*), \quad F_{pqgh}(a \circ a^*) = F_{pqgh}(a \circ a^*). \quad (3.6)$$

Also, it follows from (3.5) and the property of O as given by (3.3) that

$$\left. \begin{aligned} F_{pqgh}(a \circ a') \circ F_{pqgh}(a \circ a') &= F_{pqgh}(a \circ a') \circ O \\ F_{pqgh}(a \circ a') \circ F_{pqgh}(a \circ a') &= F_{pqgh}(a \circ a') \circ O \end{aligned} \right\} \quad (3.7)$$

For the unique element u of S which is both self-associate and self-conjugate, it follows from (3.5) that

$$F_{pqgh}(u) \circ F_{pqgh}(u) = F_{pqgh}(u) \circ O.$$

Writing $ghpq$ for $pqgh$, we get

$$F_{ghpq}(u) \circ F_{ghpq}(u) = F_{ghpq}(u) \circ O$$

And since, by (3.6), $F_{pqgh}(u) = F_{ghpq}(u)$, the above two equations are satisfied if

$$F_{pqgh}(u) = F_{ghpq}(u). \quad (3.8)$$

3.1. With reference to the correspondence between the elements $f(a)$ and $F(a)$, we may regard the properties (2.2) and (2.3) as corresponding to (3.4) and (3.5) respectively. The first of these two pairs of corresponding properties suggest the possibility of creating an element, say ϵ , such that

$$F_{ijk}(\epsilon) = O. \quad (3.9)$$

In order to examine the possibility, we notice the correspondence (3.2) and that arising from (2.6), (3.6), namely

$$f(a) = f_{ij}(a) \leftrightarrow F_{ijk}(a) = F(a)$$

$$\varphi_{ij} = f_{ij}(a \circ a^*) \circ f_{ij}(a \circ a^*) \leftrightarrow F_{ijk}(a \circ a^*) \circ F_{ijk}(a \circ a^*) = O. \quad (3.10)$$

Since f_{ij} is not generally equal to f_{ij} for the same element, it appears from (3.9) and (3.10) that an element ϵ as proposed cannot normally belong to S ; and that this element, if created, has to be self-associate. Under the circumstances, let it be possible to express φ_{ij} as

$$\varphi_{ij} = f_{ij}(\epsilon) \circ f_{ij}(\epsilon)$$

with the conditions that ϵ is self-associate and $F_{ijk}(\epsilon) = F_{ijk}(\epsilon)$. Then ϵ shall be the required element.

Let us for the moment suppose that a is any self-associate element of S . Then it follows from II(1) § 1 that

$$a_1 = a_2, \quad a_3 = a_p, \quad a_4 = a_{p-1}, \quad a_5 = a_{p-2}, \dots$$

$$\text{Or } a_j = a_{p-(j-3)}, \quad j = 1, \dots, p/2 + 1.$$

$$\therefore u = a_j \circ a_{j+p/2} = a_j \circ a_{p+p/2-(j-3)}.$$

In particular,

$$u = a_{p/4} \circ a_{p/4+3}.$$

If now the self-associate element ϵ generates a system S' in exactly the same manner as t generates S and if the unique element u is both self-associate and self-conjugate in both S and S' , then a connecting link of S and S' is given by

$$u = \epsilon_{p/4} \circ \epsilon_{p/4+3}. \quad (8.11)$$

4. Let a, b be two elements of S and $a = a_1, a_2, \dots, a_p$; $b = b_1, b_2, \dots, b_p$ be the sequences of the type (1.5) generated by them and let, as before,

$$u = a_l \circ a_{l+p/2} = b_r \circ b_{r+p/2}, \quad l, r = 1, 2, \dots \quad (4.1)$$

be the unique element of S which is both self-associate and self-conjugate.

Now the functions f and F may not be completely determined by the properties given in the last two articles. We may therefore assume for them further properties. Let then f be supposed to have the distributive property:

$$f(a \circ b) = f(a) \circ f(b) \quad (4.2)$$

This makes the correspondence $a \leftrightarrow f(a)$ isomorphic. In particular, we then have

$$f(u) = f(a_l) \circ f(a_{l+p/2}) = f(a_l) \circ f(a_{l+p/2}) = f(a_l) \circ f(a_{l+p/2}).$$

Various formulae may be derived from (2.2), (2.4), (2.5) by the assumption of (4.2).

The isomorphism between the sets of the a 's and the $f(a)$'s and the mapping $f(a) \leftrightarrow F(a)$, as introduced before, do not imply the existence of an isomorphism between the a 's (or the $f(a)$'s) and the $F(a)$'s. We may therefore assume that there exists what may be called a *pseudo-isomorphism* defined as follows:

Let Σ be the set of all elements $F(a)$, $F_{pqph}(a)$, where a runs over the elements of S , together with all the elements that are obtained by repeated composition of these elements and of those generated in this manner.

Suppose that (a, b) , (c, d) are two given non-ordered pairs of elements of S and $\lambda_1, \dots, \lambda_m$ are m arbitrary elements of S which are not necessarily all distinct. Put

$$g_m = a \circ (\lambda_1 \circ (\dots \circ (\lambda_m \circ c)) \dots), \quad h_m = b \circ (\lambda_1 \circ (\dots \circ (\lambda_m \circ d)) \dots).$$

Further suppose that μ_1, \dots, μ_n are n arbitrary elements of S which may not be distinct from one another and from the λ 's. Put

$$G_n = (F(a) \circ F(b)) \circ (F(\mu_1) \circ (\dots \circ (F(\mu_n) \circ F(c \circ d)) \dots))$$

$$H_n = F(a \circ b) \circ (F(\mu_1) \circ (\dots \circ (F(\mu_n) \circ (F(c) \circ F(d))) \dots))$$

Then, as further property of F , we assume the following:

If $g_m = h_m$ holds in S , then

$$G_{2m} = H_{2m} \text{ holds in } \Sigma. \quad (4.3)$$

And if $g_m \circ g_n = h_m \circ h_n$ holds in S , then

$$(G_{2m} \circ G_{2n}) \circ G_{m+n} = (H_{2m} \circ H_{2n}) \circ H_{m+n} \text{ holds in } \Sigma. \quad (4.4)$$

Property (4.3) can be expressed in other forms by suitably choosing the λ 's and the μ 's. For example, for $m = 1$, choose $\mu_1 = \mu_2 = c \circ d$; if then we put $\lambda_1 = c$ or d , (4.3) reduces to the following:

If $a \circ c = b \circ (c \circ d)$, or $a \circ (c \circ d) = b \circ d$, then

$$(F(a) \circ F(b)) \circ F(c \circ d) = F(a \circ b) \circ (F(c \circ d) \circ (F(c \circ d) \circ (F(c) \circ F(d)))). \quad (4.5)$$

As an important particular case, let $m = 0$. Then (4.3) reduces to the following:

If $a \circ c = b \circ d$, then

$$(F(a) \circ F(b)) \circ F(c \circ d) = (F(c) \circ F(d)) \circ F(a \circ b). \quad (4.6)$$

It therefore follows from (4.1) that

$$(F(a_l) \circ F(b_r)) \circ F(a_{l+p/2} \circ b_{r+p/2}) = (F(a_{l+p/2}) \circ F(b_{r+p/2})) \circ F(a_l \circ b_r) \quad (4.7)$$

With regard to this interesting formula, we notice that since $a_l = a_{l+p}$, $b_r = b_{r+p}$, there are $(p/2)^2$ such distinct formulae if $a \neq b$, $l \neq r$. If, however, $a = b$, $l \neq r$, there are $\frac{1}{2}[(\frac{p}{2}) - \frac{1}{2}p] = p(p-2)/4$ such distinct formulae. And if $a \neq b$, $l = r$, there are $p/2$ such distinct formulae.

4.1. Now the 'objects' s, i, j, k , with respect to which permutations have been taken, may not be all distinct. If so, it follows from (3.3) and (3.4) that

$$F_{sijk}(a) = F_{sijk}(a \circ a^*) = O.$$

The invariant element O of Σ has therefore special properties which we shall define as follows:

Two elements A and B of Σ shall be called *inverse* of one another if and only if $A \circ B = O$, and the inverse of A shall be denoted by \bar{A} . It follows $\bar{\bar{A}} = A$. Further, we shall suppose that the inverse has the following properties:

$$(1) \quad \overline{A \circ B} = \bar{A} \circ \bar{B}.$$

$$(2) \quad \text{The equation } A = B \text{ implies } A \circ B = O.$$

(3) If $F(\lambda_1) \circ (F(\lambda_2) \circ (\dots \circ (F(\lambda_m)) \dots)) = F(\mu_1) \circ (F(\mu_2) \circ (\dots \circ F(\mu_n)) \dots)$ is an equation in Σ , $m \leq n$, then the equation remains unaltered when $F(\lambda_r)$ and $F(\mu_r)$ are replaced respectively by $\bar{F}(\mu_r)$ and $(\bar{F})(\lambda_r)$, $1 \leq r \leq m$.

In view of the above properties, we have

$$O = \bar{O}, \quad F_{sikj}(a) = F_{tqjk}(a^*) = \bar{F}_{sikj}(a); \quad (4.9)$$

and the equation (4.8) can be written as follows:

$$\text{If } a \circ c = b \circ d, \text{ then}$$

$$((F(a) \circ F(b)) \circ \bar{F}(a \circ b)) \circ ((\bar{F}(c) \circ \bar{F}(d)) \circ F(c \circ d)) = O. \quad (4.9)$$

5. It may be noticed that in the equation $g_m = h_m$ of (4.8), the λ 's with the same index are placed in corresponding positions on both sides of the equation, and therefore the two λ_i may be expected to 'counterbalance' one another, so to say, for certain suitably chosen composition. Further, the two pairs of elements a, c in g_m and b, d in h_m on the two sides of the same equation have a kind of correspondence in the sense that the elements of each pair are separated from one another by the 'same number (m) of steps'.

It is therefore natural to enquire whether, given the elements a, b , it is possible to obtain c, d such that, for certain composition suitably chosen, the equation $g_m = h_m$ may be identically satisfied. If it is possible, i.e., if c, d can be created in this manner, the equation $G_{2m} = H_{2m}$ shall hold without any restriction.

Evidently, given m , the elements c, d shall have to be constructed with the help of a, b . For the purpose of this construction, it appears that we require a second composition and, for that matter, we require a system of *double composition* of which S would be a subsystem. This new system may be defined as follows:

Let the system of elements S as defined in § 1 be *extended* to a system R having the following properties:

(1) For every pair of elements a, b of R , two compositions are defined in R of which one $a \circ b$ is the commutative composition already defined in S and the other $a \wedge b$ is non-commutative, and, for every element a of R , they satisfy

$$a \circ a = a \wedge a = a.$$

(2) Every element a of R has an associate a^* and a conjugate a' belonging to R , which satisfy $a^{**} = a' = a$ as in S .

$$(3) (a \circ b)^* = a^* \circ b^*, (a \circ b)' = a' \circ b', (a \wedge b)^* = a^* \wedge b^*, (a \wedge b)' = a' \wedge b'.$$

$$(4) a \circ b = (a \wedge b) \circ (b \wedge a).$$

$$(5) (a \circ b) \circ (a \wedge b) = a.$$

In view of the properties (1), (2), (3), there is no harm to suppose that all we have said in the last few articles remains true when the elements of S are replaced by the elements of R .

Correspondingly, a set Ω may be defined of which Σ is a subset such that Ω corresponds to R in the same manner as Σ corresponds to S .

Now let a, b be two arbitrary elements of R . Put

$$\left. \begin{aligned} e_1 &= a \wedge b, \quad \widehat{e}_1 = b \wedge a, \quad e_2 = e_1 \wedge \widehat{e}_1, \quad \widehat{e}_2 = \widehat{e}_1 \wedge e_1, \dots, \\ e_{m+1} &= e_m \wedge \widehat{e}_m, \quad \widehat{e}_{m+1} = \widehat{e}_m \wedge e_m, \dots \end{aligned} \right\} \quad (5.1)$$

Then it follows from the property (4) above that

$$e_m \circ \widehat{e}_m = a \circ b, \quad m = 1, 2, \dots \quad (5.2)$$

Now, given the commutative composition, it is seen from the properties (4) and (5) that the non-commutative composition has been made to depend, so to say, on the commutative one. It may then be possible to choose the commutative composition in such a manner that the following equation in R

$$a \circ (\lambda_1 \circ (\dots \circ (\lambda_m \circ \widehat{e}_m)) \dots) = b \circ (\lambda_1 \circ (\dots \circ (\lambda_m \circ e_m)) \dots) \quad (5.3)$$

is satisfied identically. If so, it would follow from (4.3), (5.1) and (5.2) that

$$\begin{aligned} (F(a) \circ F(b)) \circ (F(\mu_1) \circ (\dots \circ (F(\mu_{2m}) \circ F(a \circ b)) \dots)) \\ = F(a \circ b) \circ (F(\mu_1) \circ (\dots \circ (F(\mu_{2m}) \circ F(e_m) \circ F(\widehat{e}_m)) \dots)) \end{aligned} \quad (5.4)$$

holds in Ω . This formula may be put in a simpler form by choosing $\mu_1 = \dots = \mu_{2m} = a \circ b$. In particular, for $m = 1$, we may obtain in Ω

$$F(a) \circ F(b) = F(a \circ b) \circ (F(a \circ b) \circ (F(e_1) \circ F(\widehat{e}_1))). \quad (5.5)$$

Finally, if the commutative composition can be so chosen that (5.3) is identically satisfied, then (4.4), when G_{2m} and H_{2m} are expressed by the two sides of (5.4), would be satisfied in Ω without any restriction.

6. We shall now apply the properties of the algebraic system, obtained so far, in Riemannian geometry. In what follows, the usual notations of tensor calculus are adopted. Suppose that in a Riemannian space whose metric is given by

$$ds^2 = g_{ij} dx^i dx^j, \quad (6.1)$$

contravariant vectors are given parallel displacements according to the law

$$dV^i + \Gamma_{ij}^i V^j dx^j = 0, \quad (6.2)$$

where the coefficients of affine connection Γ_{ij}^i are supposed to be arbitrary. We give below a brief summary of those results, obtained in the paper referred to (Sen, 1950), which are here necessary for our purpose.

Let the covariant derivatives of tensors with respect to (6.2) be denoted by a comma followed by indices. If then we denote and define as follows

$$a = \Gamma_{ij}^t, \quad a^* = \Gamma_{ij}^t + g^{mt} g_{im,j}, \quad a' = \Gamma_{ij}^t, \quad (6.3)$$

the property (1.1) is satisfied. Also, if the commutative composition of two affine connections in the space denoted by a and b is defined as

$$a \circ b = \frac{1}{2}(a + b), \quad (6.4)$$

the properties (1.2) and (1.5) are satisfied. We can therefore construct the system S of affine connections generated by a as envisaged in § 1.

Further, if we put

$$a = a_1, \quad a^* = a_2, \quad a^{*'} = a_3, \quad a^{*''} = a_4, \dots,$$

$$\alpha = g^{mt} g_{im,j}, \quad \alpha_o = g^{mt} g_{jm,i}, \quad \beta = g^{mt} g_{in}(\Gamma_{mj}^n - \Gamma_{jn}^m), \quad \beta_o = g^{mt} g_{jn}(\Gamma_{mi}^n - \Gamma_{im}^n), \quad \gamma = g^{mt} g_{ij,m} = \gamma_o,$$

we obtain the following cyclic sequence of the type (1.5) with 12 elements

$$\left. \begin{aligned} a_1, \quad a_2 = a + \alpha, \quad a_3 = a' + \alpha_o, \quad a_4 = a + \alpha + \beta - \gamma, \quad a_5 = a' + \alpha_o + \beta_o - \gamma \\ a_6 = a + \alpha + \alpha_o + \beta + \beta_o - \gamma, \quad a_7 = a' + \alpha + \alpha_o + \beta + \beta_o - \gamma, \quad a_8 = a' + \alpha_o + \beta + \beta_o - \gamma \\ a_9 = a + \alpha + \beta + \beta_o - \gamma, \quad a_{10} = a' + \alpha_o + \beta_o, \quad a_{11} = a + \alpha + \beta, \quad a_{12} = a' \end{aligned} \right\} \quad (6.5)$$

Finally, if $\left\{ \begin{smallmatrix} t \\ ij \end{smallmatrix} \right\}$ denotes the Christoffel symbol, then, corresponding to (1.6), we have

$$\left\{ \begin{smallmatrix} t \\ ij \end{smallmatrix} \right\} = \frac{1}{2}(a_i + a_{i+j}), \quad i = 1, \dots, 6. \quad (6.6)$$

as the unique element of S which is both self-associate and self-conjugate.

Now put

$$f(a) = g_{st} \Gamma_{ij}^t = f_{stij}(a) \quad (6.7)$$

Then

$$f_{stij}(a) + f_{stij}(a^*) = g_{st} \Gamma_{ij}^t + g_{st}(\Gamma_{ij}^t + g^{mt} g_{im,j}) = g_{st,j} + g_{st} \Gamma_{ij}^t + g_{st} \Gamma_{sj}^t = \frac{\partial g_{st}}{\partial x^j}. \quad (6.8)$$

Further, it may be verified from (6.5) that

$$\begin{aligned} f_{stij}(a) + f_{jits}(a^{*'}) &= g_{st} \Gamma_{ij}^t + g_{jt} [\Gamma_{si}^t + g^{mt} \{g_{sm,i} + g_{sn}(\Gamma_{mi}^n - \Gamma_{im}^n)\}] \\ &= g_{jst,i} + g_{st} \Gamma_{ji}^t + g_{jt} \Gamma_{si}^t = \frac{\partial g_{js}}{\partial x^i}. \end{aligned}$$

Similarly

$$f_{jits}(a) + f_{stij}(a^{*'}) = \frac{\partial g_{st}}{\partial x^i}.$$

Thus, (2.2) and (2.3) are satisfied and therefore all subsequent equations of § 2 are satisfied, as is to be expected.

Further, put

$$F(a) = g_{st} \Gamma_{ijk}^t = g_{st} \left(\frac{\partial \Gamma_{ik}^t}{\partial x^j} - \frac{\partial \Gamma_{ij}^t}{\partial x^k} + \Gamma_{mj}^t \Gamma_{ik}^m - \Gamma_{mk}^t \Gamma_{ij}^m \right) = \Gamma_{stijk} = F_{stijk}(a). \quad (6.9)$$

Then

$$\begin{aligned}\Gamma_{sijk} &= \frac{\partial g_{sk} \Gamma_{ij}^t}{\partial x^j} - \frac{\partial g_{st} \Gamma_{ij}^t}{\partial x^k} - \Gamma_{ik}^t \left(\frac{\partial g_{st}}{\partial x^j} - g_{sm} \Gamma_{ij}^m \right) + \Gamma_{ij}^t \left(\frac{\partial g_{st}}{\partial x^k} - g_{sm} \Gamma_{ik}^m \right) \\ &= \frac{\partial g_{sk} \Gamma_{ij}^t}{\partial x^j} - \frac{\partial g_{st} \Gamma_{ij}^t}{\partial x^k} - (g_{st,j} \Gamma_{ik}^t - g_{st,k} \Gamma_{ij}^t) - g_{tm} (\Gamma_{ik}^m \Gamma_{sj}^t - \Gamma_{ij}^t \Gamma_{sk}^m).\end{aligned}$$

Therefore

$$\begin{aligned}\Gamma_{sijk} + \Gamma_{tsjk} &= \frac{\partial}{\partial x^j} (g_{st} \Gamma_{ik}^t + g_{it} \Gamma_{sk}^t) - \frac{\partial}{\partial x^k} (g_{st} \Gamma_{ij}^t + g_{it} \Gamma_{sj}^t) \\ &\quad + g_{sk,k} \Gamma_{ij}^t + g_{it,k} \Gamma_{sj}^t - g_{st,j} \Gamma_{ik}^t - g_{it,j} \Gamma_{sk}^t. \\ &= \frac{\partial}{\partial x^j} (g_{is,k} + g_{st} \Gamma_{ik}^t + g_{it} \Gamma_{sk}^t) - \frac{\partial}{\partial x^k} (g_{is,j} + g_{st} \Gamma_{ij}^t + g_{it} \Gamma_{sj}^t) \\ &\quad + g_{is,jk} - g_{is,kj} + g_{ist} (\Gamma_{jk}^t - \Gamma_{kj}^t).\end{aligned}$$

Therefore

$$\Gamma_{sijk} + \Gamma_{tsjk} = g_{is,jk} - g_{is,kj} + g_{ist} (\Gamma_{jk}^t - \Gamma_{kj}^t). \quad (6.10)$$

And, if we put $F(a^*) = \Gamma_{sijk}^*$ we have (Sen, 1948)

$$\Gamma_{sijk} - \Gamma_{stjk}^* = g_{is,jk} - g_{is,kj} + g_{ist} (\Gamma_{jk}^t - \Gamma_{kj}^t). \quad (6.11)$$

It therefore follows from (6.10) and (6.11) that (interchanging i, s)

$$\Gamma_{sijk} + \Gamma_{istj}^* = 0. \quad (6.12)$$

Thus (3.3), (3.4) and (3.5) are satisfied when O is replaced here by the number zero. Therefore, the subsequent equations (3.6) to (3.8) of § 3 are also satisfied, as is to be expected.

The equation (6.12) is interesting. It says that Γ_{sijk} is skew in the indices s, i for those parallelism which are self-associate. It explains therefore as to why the Riemann-Christoffel tensor K_{sijk} has this property. The same thing is true of the property (3.6). Evidently, K_{sijk} is not the only one of its kind which has these two properties. But the property (3.8), namely $\Gamma_{sijk} = \Gamma_{jkst}$, holds only for the Riemann-Christoffel tensor K_{sijk} and for no other tensor of its kind.

Regarding the equation (3.9) and the remarks made thereabout, it follows from (6.8) that

$$\varphi_{sij} = \frac{1}{2} \{f_{sij}(\epsilon) + f_{ijs}(\epsilon)\} = \frac{1}{2} \frac{\partial g_{is}}{\partial x^j}. \quad (6.13)$$

Let us introduce an orthogonal ennuple at every point of the space defined, as usual, by

$$g_{is} = \sum_T \tau h_i^T \tau h_s^T, \quad \tau h^i = g^{im} \tau h_m^T.$$

Then

$$\frac{\partial g_{is}}{\partial x^j} = \sum_T \left(\tau h_s^T \frac{\partial \tau h_i^T}{\partial x^j} + \tau h_i^T \frac{\partial \tau h_s^T}{\partial x^j} \right) = \sum_T \left(g_{st} \tau h^t \frac{\partial \tau h_i^T}{\partial x^j} + g_{it} \tau h^t \frac{\partial \tau h_s^T}{\partial x^j} \right).$$

If therefore we put $\epsilon = \tau h^t (\partial \tau h_i / \partial x^j)$ then (6.13) is satisfied in view of (6.7). Now the

parallelism (6.2) corresponding to $\Gamma_{ij}^t = \epsilon$ is Einstein's teleparallelism which is known to be self-associate and with respect to which the curvature tensor is known to vanish for all permutations of the indices

7. Let a and b be two elements of S and let

$$a = \Gamma_{ij}^t, \quad b = L_{ij}^t, \quad \frac{1}{2}(a+b) = \Lambda_{ij}^t, \quad a-b = T_{ij}^t.$$

Then it can be seen (Sen, 1949) that

$$F(\frac{1}{2}(a+b)) - \frac{1}{2}\{F(a) + F(b)\} \equiv \Lambda_{sijk} - \frac{1}{2}(\Gamma_{sijk} + L_{sijk}) = \frac{1}{2}g_{st}(T_{mk}^t T_{ij}^m - T_{mj}^t T_{ik}^m). \quad (7.1)$$

If c and d are two other elements of S such that $|a-b| = v|c-d|$, where v is any constant multiplier, it follows from (7.1) that

$$F(\frac{1}{2}(a+b)) - \frac{1}{2}\{F(a) + F(b)\} = v^2[F(\frac{1}{2}(c+d)) - \frac{1}{2}\{F(c) + F(d)\}]. \quad (7.2)$$

In view of the composition as defined by (3.4), repeated composition introduces the numbers $v = 1/2^t$, where t is an integer. We may therefore verify from (7.2) that (4.3) and (4.4) are satisfied. Therefore the subsequent equations of § 4 are also satisfied.

Some applications of (4.3), (4.4) were given in the paper (Sen, 1949) referred to above. For example, it was shown there that the following relation existed among the four elements of S :

$$\varphi_{ij}^t - \theta_{ij}^t = \frac{1}{2} \left(\nabla_{ij}^t - \left\{ \frac{t}{ij} \right\} \right).$$

Therefore, putting

$$\chi_{ij}^t = \frac{1}{2}(\varphi_{ij}^t + \theta_{ij}^t), \quad \psi_{ij}^t = \frac{1}{2} \left(\nabla_{ij}^t + \left\{ \frac{t}{ij} \right\} \right),$$

we obtained the following formula, as a particular case of (7.2),

$$K_{ijk}^t + \nabla_{ijk}^t - 4\theta_{ijk}^t - 4\varphi_{ijk}^t - 2\psi_{ijk}^t + 8\chi_{ijk}^t = 0.$$

The equation (4.7) with $a = b$, $l \neq r$ gives

$$F(\frac{1}{2}(a_l + a_r)) - \frac{1}{2}\{F(a_l) + F(a_r)\} = F(\frac{1}{2}(a_{l+s} + a_{r+s})) - \frac{1}{2}\{F(a_{l+s}) + F(a_{r+s})\}. \quad (7.3)$$

Again put

$$F(a^*) = \Gamma_{sijk}^*, \quad F(a^{*'}) = \Gamma_{sijk}^{*'}, \quad F(a^{**}) = \Gamma_{sijk}^{**}, \quad \dots, \quad F(a') = \Gamma_{sijk}^{'}$$

where

$$\Gamma_{sijk}^* = g_{st}\Gamma_{ijk}^t, \quad \dots, \quad \Gamma_{sijk}^{'} = g_{st}\Gamma_{ijk}^t, \quad \Gamma_{ijk}^t = \frac{\partial \Gamma_{ki}^t}{\partial x^j} - \frac{\partial \Gamma_{ji}^t}{\partial x^k} + \Gamma_{jm}^t \Gamma_{ik}^m - \Gamma_{km}^t \Gamma_{ji}^m,$$

and similarly for the b 's and the $\frac{1}{2}(a+b)$'s. Also put

$$A_1 = \Lambda_{sijk} - \frac{1}{2}(\Gamma_{sijk} + L_{sijk}), \quad A_2 = \Lambda_{sijk}^* - \frac{1}{2}(\Gamma_{sijk}^* + L_{sijk}^*),$$

$$A_3 = \Lambda_{sijk}^{*'} - \frac{1}{2}(\Gamma_{sijk}^{*'} + L_{sijk}^{*'}), \quad \dots, \quad A_{12} = \Lambda_{sijk}^{'} - \frac{1}{2}(\Gamma_{sijk}^{'} + L_{sijk}^{'}).$$

Then the equation (4.7), with $a \neq b$, $l = r$ gives the interesting relations

$$A_1 = A_7, \quad A_2 = A_8, \quad \dots, \quad A_6 = A_{12}. \quad (7.4)$$

7.1. Regarding the non-commutative composition introduced in § 5, let us put

$$a \wedge b = \frac{1}{2}(a+b) + (a-b) = \Lambda_{ij}^t + T_{ij}^t = W_{ij}^t. \quad (7.5)$$

Evidently, W_i^t are the coefficients of an affine connection. Let the covariant derivatives with respect to the parallelism (3.2) corresponding to Γ_{ij}^t , L_{ij}^t , Λ_{ij}^t and W_i^t be denoted respectively by a comma, a solidus, an ordinary bracket and a square bracket followed by indices. Then

$$[g_{ij}]_k = (g_{ij})_k - g_{jt} T_{ik}^t - g_{it} T_{jk}^t$$

$$\therefore (a \wedge b)^* = W_{ij}^t + g^{mt} [g_{im}]_j = \Lambda_{ij}^t + g^{mt} (g_{im})_j - g^{mt} g_{in} T_{mk}^n$$

Now

$$\frac{1}{2}(a^* + b^*) = \Lambda_{ij}^t + g^{mt} (g_{im})_j$$

and

$$(a^* - b^*) = T_{ij}^t + g^{mt} (g_{im})_j - g_{im} T_{jn}^n = T_{ij}^t - g^{mt} (g_{mn} \Gamma_{ij}^n + g_{jn} T_{mi}^n) = -g^{mt} g_{in} T_{mj}^n$$

Hence

$$[\frac{1}{2}(a+b) + (a-b)]^* = \frac{1}{2}(a^* + b^*) + (a^* - b^*). \quad (7.6)$$

Therefore all the properties (1) to (5) of the system R are satisfied.

As regards (5.1), we notice that

$$e_{m+1} = \frac{1}{2}(a+b) + 2^m(a-b), \quad \widehat{e}_{m-1} = \frac{1}{2}(a+b) + 2^m(b-a).$$

It can therefore be verified that (5.3) is satisfied and therefore (5.4) is satisfied. In particular, formula (5.5) becomes

$$4\{F(a) + F(b)\} = 6F(\frac{1}{2}(a+b)) + F(\frac{1}{2}(a+b) + (a-b)) + F(\frac{1}{2}(a+b) + (b-a)) \quad (7.7)$$

It is thus shown that various new and interesting formulae can be obtained in Riemannian geometry and that their origin may be traced to the kind of abstract algebra discussed in the paper.

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CORRECTIONS TO MY PAPER "ON THE FLEXURE PROBLEM OF A LIMACON AND SOME OTHER BOUNDARIES."

BY
D. N. MITRA

The following error has crept into my paper with the above title published in *Bull. Cal. Math. Soc.* **14**, 153, 1949.

Page 157, 5th line from the top, for $x_f = \frac{-a^5b^2(3+4\sigma)}{(1+\sigma)(b^2+2a^2)(b^4+6a^2b^2+2a^4)}$

$$\text{read } x_f = \frac{-2a^5b^2(3+4\sigma)}{(1+\sigma)(b^2+2a^2)(b^4+6a^2b^2+2a^4)}$$

I am thankful to the reviewer of my paper in the "Mathematical Reviews" (Vol. 11 p. 287) for pointing out this misprint (omission of factor 2).

I take this opportunity in clarifying a statement in my paper to the effect that the error in Stevenson's final result is due to an unnecessary factor 2 in equation (3.26) of Stevenson's paper. I did not mean that Stevenson's result differs from mine by a factor 2. In fact Stevenson's result in our notations is,

$$x_f = \frac{-2ab^2\{(b^2-a^2)^2+\sigma(b^4-2a^2b^2+4a^4)\}}{3(1+\sigma)(b^2+2a^2)(b^4+6a^2b^2+2a^4)}.$$

This is completely different from the result I have obtained.

BENDING OF ANNULAR ELLIPTICAL PLATES LOADED BY EDGE MOMENTS.

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(*Communicated by the Secretary—Received April 24, 1960*)

Summary

The problem of the small deflections of a thin plate bounded by two confocal ellipses and loaded by bending moments around both boundaries is presented here. Three conditions of edge support are considered. Since the faces are free of load, the Lagrange differential equation for the middle surface of the plate reduces to the biharmonic equation. Elliptic coordinates are employed and a series solution of the biharmonic equation is used to represent the deflection surface of the plate. Coefficients of the various terms in the series are obtained merely by solving linear, algebraic equations.

Nomenclature

- c = distance from center of ellipse to either focus
- D = flexural rigidity of plate = $\frac{Et^3}{12(1-\nu^2)}$
- E = Young's modulus
- ξ, η = elliptic coordinate directions
- M_x, M_y = bending moments, per unit length of the middle surface of the plate, in rectangular coordinates
- M_{xy}, M_{yx} = twisting moments, per unit length of the middle surface of the plate, in rectangular coordinates
- Q_x, Q_y = shearing forces, per unit length of the middle surface of the plate, parallel to the x -axis
- M_ξ, M_η = bending moments, per unit length of the middle surface of the plate, in elliptic coordinates
- $M_{\xi\eta}, M_{\eta\xi}$ = twisting moments, per unit length of the middle surface of the plate, in elliptic coordinates
- Q_ξ, Q_η = shearing forces, per unit length of the middle surface of the plate, parallel to the ξ -axis
- q = normal load per unit area acting on face of plate
- t = thickness of plate
- w = z -component of displacement of a particle originally in the middle plane of the plate
- ν = Poisson's ratio

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Introduction

The literature contains the solutions of only a relatively few elasticity problems involving elliptical boundaries. Most of these may be found in the references at the end of this paper. The problem treated here may be regarded as an extension of the work included in a previous paper by the author (Nash, 1950). The solution of any plane stress problem involving elliptical boundaries should be evident from a consideration of the general theory presented here.

Theory developed here is subject to the following restrictions :

- (1) The plate is composed of a homogeneous, isotropic material.
- (2) The material follows Hooke's Law
- (3) The deflection of the plate is small compared to the thickness
- (4) The thickness of the plate is small compared to its lateral dimensions.

Deflections. The differential equation in rectangular coordinates of the middle surface of a plate is

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \quad (1)$$

where q is the normal load per unit area acting on the face of the plate and D is the flexural rigidity of the plate. Here, w denotes the z -component of displacement of a particle originally in the middle plane of the plate. The quantity w is called the deflection of the plate. This equation may be written in the form

$$\nabla^4 w = \frac{q}{D} \quad (2)$$

where ∇^2 is the Laplacian operator

Since the problem under consideration here involves a plate of elliptical contour, elliptic coordinates will be used. The equations relating rectangular Cartesian coordinates (x, y) to elliptic coordinates (ξ, η) are

$$\left. \begin{aligned} x &= c \cosh \xi \cos \eta \\ y &= c \sinh \xi \sin \eta \end{aligned} \right\} \quad (3)$$

The parametric lines of the elliptic coordinate system are the orthogonal families of confocal ellipses and hyperbolas shown in Fig. 1.

By means of equations (3), the Laplacian operator may be expressed in terms of elliptic coordinates. It is

$$\nabla^2 = \frac{2}{c^2(\cosh 2\xi - \cos 2\eta)} \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \quad (4)$$

Thus, the differential equation of the middle surface of the plate, expressed in elliptic coordinates, is

$$\left\{ \frac{2}{c^2(\cosh 2\xi - \cos 2\eta)} \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \right\}^2 w = \frac{q}{D} \quad (5)$$

The solution of the problem of the bending of a plate of elliptical contour when a resultant normal load q per unit area acts on the faces of the plate and the edges are loaded by bending moments, twisting moments, and shearing forces thus reduces to the integration of Equation (6) with regard to boundary conditions.

Moments. The expressions in rectangular coordinates for bending and twisting moments, per unit length of the middle surface of the plate, are

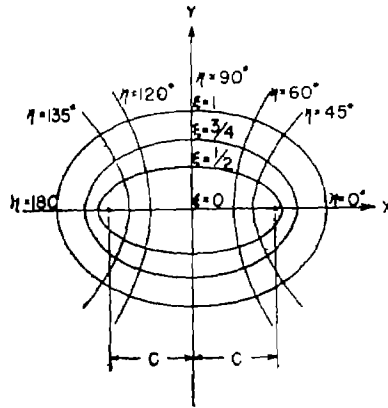


Fig. 1. The elliptic coordinate system

$$M_x = -D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \quad (6)$$

$$M_y = -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \quad (7)$$

$$M_{xy} = -M_{yx} = D(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \quad (8)$$

where ν is Poisson's ratio.

In the elliptic coordinate system, the bending moments, per unit length of the middle surface of the plate, are denoted by M_ξ and M_η , and the twisting moments, per unit length of the middle surface by M_ξ and M_η . The shearing forces, also per unit length of the middle surface, parallel to the z -axis are denoted by Q_ξ and Q_η . By a consideration of the equilibrium of the stresses acting on an element bounded by an ellipse and lines parallel to the x and y coordinate axes, the following expressions are obtained:

$$M_\xi = -\frac{2D}{c^2(\cosh 2\xi - \cos 2\eta)} \left[\frac{\partial^2 w}{\partial \xi^2} + \nu \frac{\partial^2 w}{\partial \eta^2} - \frac{(1-\nu) \sinh 2\xi}{(\cosh 2\xi - \cos 2\eta)} \cdot \frac{\partial w}{\partial \xi} + \frac{(1-\nu) \sin 2\eta}{(\cosh 2\xi - \cos 2\eta)} \cdot \frac{\partial w}{\partial \eta} \right], \quad (9)$$

$$M_{\eta} = -\frac{2D}{c^2(\cosh 2\xi - \cos 2\eta)} \left[\nu \cdot \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} + \frac{(1-\nu) \sinh 2\xi}{(\cosh 2\xi - \cos 2\eta)} \cdot \frac{\partial w}{\partial \xi} - \frac{(1-\nu) \sin 2\eta}{(\cosh 2\xi - \cos 2\eta)} \cdot \frac{\partial w}{\partial \eta} \right], \quad (10)$$

$$M_{\xi\eta} = \frac{D(1-\nu)}{c^2(\cosh 2\xi - \cos 2\eta)} \left[\frac{\sin 2\eta}{(\cosh 2\xi - \cos 2\eta)} \cdot \frac{\partial w}{\partial \xi} + \frac{\sinh 2\xi}{(\cosh 2\xi - \cos 2\eta)} \cdot \frac{\partial w}{\partial \eta} - \frac{\partial^2 w}{\partial \xi \partial \eta} \right]. \quad (11)$$

Solution of the Biharmonic Equation. The differential equation of the middle surface of a plate has already been found in elliptic coordinates for the case when a resultant normal load q per unit area acts on the faces of the plate and the edges are loaded by bending moments, twisting moments, and shearing forces. This is Equation (5).

If the external forces acting on the plate are applied only to the edges, the faces being free, then $q = 0$, and the foregoing equation reduces to

$$\left\{ \frac{1}{(\cosh 2\xi - \cos 2\eta)} \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \right\}^2 w = 0 \quad (12)$$

This is the biharmonic equation expressed in elliptic coordinates.

In a previous paper (Nash, 1950) the author has found the following solutions to this equation. They are of two general types:

(a) Those that are harmonic, i.e., they satisfy Laplace's equation. Such functions are:

$$\left. \begin{aligned} &1, \xi, \eta, \\ &\sinh n\xi \sin n\eta, \sinh n\xi \cos n\eta, \\ &\cosh n\xi \sin n\eta, \cosh n\xi \cos n\eta, \end{aligned} \right\} n = 1, 2, \dots \quad (13)$$

(b) Those that are biharmonic, but not harmonic. Such functions are:

$$\left. \begin{aligned} &\sinh (n+2)\xi \sin n\eta + \sinh n\xi \sin (n+2)\eta, \quad n = 1, 2, \dots \\ &\sinh (n+2)\xi \cos n\eta + \sinh n\xi \cos (n+2)\eta, \quad n = 1, 2, \dots \\ &\cosh (n+2)\xi \sin n\eta + \cosh n\xi \sin (n+2)\eta, \quad n = 1, 2, \dots \\ &\cosh (n+2)\xi \cos n\eta + \cosh n\xi \cos (n+2)\eta, \quad n = 0, 1, 2, \dots \end{aligned} \right\} \quad (14)$$

The First Problem

Statement of the Problem. Consider a plate bounded by two confocal ellipses and subject to the following boundary conditions:

(a) The inner edge of the plate, $\xi = \xi_1$, is supported and is given a prescribed deflection in a direction perpendicular to the middle plane of the plate.

(b) The outer edge of the plate, $\xi = \xi_2$, is supported and is given a prescribed deflection in a direction perpendicular to the middle plane of the plate.

(c) A general distribution of bending moments is applied to the inner edge of the plate. Let M_1 denote this moment at $\xi = \xi_1$.

(d) A general distribution of bending moments is applied to the outer edge of the plate. Let M_2 denote this moment at $\xi = \xi_2$.

Let us assume for the deflection $w(\xi, \eta)$ the following linear combination of biharmonic functions:

$$\begin{aligned}
 w = & \sum_{n=3}^{\infty} A_n \sinh n\xi \sin n\eta + \sum_{n=2}^{\infty} B_n \sinh n\xi \cos n\eta + \sum_{n=3}^{\infty} C_n \cosh n\xi \sin n\eta \\
 & + \sum_{n=2}^{\infty} D_n \cosh n\xi \cos n\eta + E_0 (\sinh 2\xi + \sin 2\eta) \\
 & + \sum_{n=1}^{\infty} E_n [\sinh (n+2)\xi \sin n\eta + \sinh n\xi \sin (n+2)\eta] \\
 & + \sum_{n=1}^{\infty} F_n [\sinh (n+2)\xi \cos n\eta + \sinh n\xi \cos (n+2)\eta] \\
 & + \sum_{n=1}^{\infty} G_n [\cosh (n+2)\xi \sin n\eta + \cosh n\xi \sin (n+2)\eta] \\
 & + \sum_{n=0}^{\infty} H_n [\cosh (n+2)\xi \cos n\eta + \cosh n\xi \cos (n+2)\eta]
 \end{aligned} \tag{15}$$

where A_n, B_n, \dots, H_n are constants to be determined from the boundary conditions.

The First Boundary Condition. The prescribed deflection of any point on the neutral axis of the inner edge of the plate may be represented by the Fourier series

$$[w]_{\xi=\xi_1} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\eta + b_n \sin n\eta) \tag{16}$$

where a_0, a_n , and b_n are known constants. This boundary condition may be applied to the value of w given by Equation (15) to yield an identity in η . Coefficients of corresponding sine and cosine terms in this identity are then equated to give one infinite set of equations containing the B_n, D_n, F_n and H_n ; and another infinite set of equations containing the A_n, C_n, E_n , and G_n .

The Second Boundary Condition. In an analogous manner, the prescribed deflection of any point on the neutral axis of the outer edge of the plate may be represented by the Fourier series

$$[w]_{\xi=\xi_2} = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} (c_n \cos n\eta + d_n \sin n\eta) \tag{17}$$

where c_0, c_n , and d_n are known constants. Again, this condition may be applied to Equation (15) to yield another identity in η . Equating coefficients of corresponding sine and cosine terms yields one infinite set of equations containing the B_n, D_n, F_n and H_n ; and another infinite set of equations containing the A_n, C_n, E_n and G_n .

The Third Boundary Condition. The expression for the bending moment M_ξ at any point in an elliptical plate is given by Equation (9). Let M_1 denote this applied moment at the inner edge $\xi = \xi_1$. We then have

$$-\frac{M_1 c^2 (\cosh 2\xi_1 - \cos 2\eta)^2}{2D} = \left\{ \left[\frac{\partial^2 w}{\partial \xi^2} \right]_{\xi=\xi_1} + \nu \left[\frac{\partial^2 w}{\partial \eta^2} \right]_{\xi=\xi_1} \right\} \cdot (\cosh 2\xi_1 - \cos 2\eta) - (1-\nu) \sinh 2\xi_1 \left[\frac{\partial w}{\partial \xi} \right]_{\xi=\xi_1} + (1-\nu) \sin 2\eta \left[\frac{\partial w}{\partial \eta} \right]_{\xi=\xi_1} \quad (18)$$

Also, M_1 may be expanded in a Fourier series of the form

$$-\frac{M_1 c^2 (\cosh 2\xi_1 - \cos 2\eta)^2}{2D} = \frac{1}{2}e_0 + \sum_{n=1}^{\infty} (e_n \cos n\eta + f_n \sin n\eta) \quad (19)$$

where e_0 , e_n , and f_n are known Fourier coefficients. This boundary condition may be applied to the value of w given by Equation (15) to yield a third identity in η . Two more infinite sets of equations, one containing the B_n , D_n , F_n , and H_n and the other A_n , C_n , E_n , and G_n are obtained by equating coefficients of corresponding sine and cosine terms.

The Fourth Boundary Condition If M_2 denotes the applied moment at the outer edge, then we may write an expression analogous to Equation (18) but with M_1 replaced by M_2 and ξ_1 replaced by ξ_2 . The left side of this equation may be expanded in a Fourier series of the form

$$-\frac{M_2 c^2 (\cosh 2\xi_2 - \cos 2\eta)^2}{2D} = \frac{1}{2}g_0 + \sum_{n=1}^{\infty} (g_n \cos n\eta + h_n \sin n\eta) \quad (20)$$

where g_0 , g_n , and h_n are known Fourier coefficients. Again, this condition may be applied to Equation (15) to give a fourth identity in η . Two more infinite sets of equations, one containing the B_n , D_n , F_n and H_n , and the other the A_n , C_n , E_n , and G_n are obtained by equating coefficients of corresponding sine and cosine terms.

Determination of Coefficients. The eight sets of equations resulting from the boundary conditions may be grouped in the following manner:

$$\left. \begin{aligned} E_0 S'_{12} + H_0 T'_{12} &= \frac{1}{2}a_0 \\ E_0 X'_{12} + H_0 Y'_{12} &= \frac{1}{2}c_0 \end{aligned} \right\} \quad (21)$$

$$\left. \begin{aligned} F_1 S'_{18} + H_1 T'_{13} &= a_1 \\ F_1 X'_{13} + H_1 Y'_{18} &= c_1 \end{aligned} \right\} \quad (22)$$

$$\left. \begin{aligned} B_2 S'_{12} + D_2 T'_{12} + F_2 S'_{14} + H_2 T'_{14} + H_0 &= a_2 \\ B_2 X'_{12} + D_2 Y'_{12} + F_2 X'_{14} + H_2 Y'_{14} + H_0 &= c_2 \\ B_2 P_{30} + D_2 P_{00} + F_2 P_{10,0} + H_2 P_{14,0} + H_0 P_{13,0} + E_0 P_{90} &= \frac{1}{2}e_0 \\ B_2 U_{30} + D_2 U_{00} + F_2 U_{10,0} + H_2 U_{14,0} + H_0 U_{13,0} + E_0 U_{90} &= \frac{1}{2}g_0 \end{aligned} \right\} \quad (23)$$

$$\left. \begin{aligned}
 & B_n S'_{1n} + D_n T'_{1n} + F_n S'_{1,n+2} + F_{n-2} S'_{1,n-2} + H_n T'_{1,n+2} + H_{n-2} T'_{1,n-2} = a_n \\
 & B_n X'_{1n} + D_n Y'_{1n} + F_n X'_{1,n+2} + F_{n-2} X'_{1,n-2} + H_n Y'_{1,n+2} + H_{n-2} Y'_{1,n-2} = c_n \\
 & B_{n-4} P_{1,n-2} + B_{n-2} P_{2,n-2} + B_n P_{3,n-2} + D_{n-4} P_{4,n-2} \\
 & \quad + D_{n-2} P_{5,n-2} + D_n P_{6,n-2} + F_{n-6} P_{7,n-2} + F_{n-4} P_{8,n-2} \\
 & \quad + F_{n-2} P_{9,n-2} + F_n P_{10,n-2} + H_{n-6} P_{11,n-2} + H_{n-4} P_{12,n-2} \\
 & \quad + H_{n-2} P_{13,n-2} + H_n P_{14,n-2} = e_{n-2} \\
 & B_{n-4} U_{1,n-2} + B_{n-2} U_{2,n-2} + B_n U_{3,n-2} + D_{n-4} U_{4,n-2} \\
 & \quad + D_{n-2} U_{5,n-2} + D_n U_{6,n-2} + F_{n-6} U_{7,n-2} + F_{n-4} U_{8,n-2} \\
 & \quad + F_{n-2} U_{9,n-2} + F_n U_{10,n-2} + H_{n-6} U_{11,n-2} \\
 & \quad + H_{n-4} U_{12,n-2} + H_{n-2} U_{13,n-2} + H_n U_{14,n-2} = g_{n-2}, \quad n = 3, 4, \dots
 \end{aligned} \right\} \quad (24)$$

$$\left. \begin{aligned}
 & E_1 S'_{13} + G_1 T'_{13} = b_1 \\
 & E_1 X'_{13} + G_1 Y'_{13} = d_1
 \end{aligned} \right\} \quad (25)$$

$$\left. \begin{aligned}
 & E_2 S'_{14} + G_2 T'_{14} + E_0 = b_2 \\
 & E_2 Y'_{14} + G_2 Y'_{14} + E_0 = d_2
 \end{aligned} \right\} \quad (26)$$

The remaining equations for the A_n , C_n , E_n , and G_n are analogous to Equations (24) with the B_n in each equation replaced by A_n , the D_n by C_n , the F_n by E_n , and the H_n by G_n . Also, on the right side of each of these equations the a_n must be replaced by b_n , the c_n by d_n , the e_n by f_n and the g_n by h_n . In Equations (24) the value zero is to be assigned to any coefficients outside of the range of definition indicated in Equation (15). In these equations, P_{mn} are known constants given by the relations:

$$\begin{aligned}
 P_{1n} &= R_2 S_{1,n-2} + R_4 S_{3,n-2} + R_6 S_{5,n-2} \\
 P_{2n} &= R_1 T_{1n} + R_3 S_{2n} + R_5 S_{2n} \\
 P_{3n} &= -R_2 S_{1,n+2} + R_4 S_{3,n+2} + R_6 S_{5,n+2} \\
 P_{4n} &= R_2 T_{1,n-2} + R_4 T_{3,n-2} + R_6 T_{5,n-2} \\
 P_{5n} &= R_1 S_{1n} + R_3 T_{2n} + R_5 T_{2n} \\
 P_{6n} &= -R_2 T_{1,n+2} + R_4 T_{3,n+2} + R_6 T_{5,n+2} \\
 P_{7n} &= R_2 S_{3,n-2} + R_4 S_{5,n-2} + R_6 S_{7,n-2}
 \end{aligned} \quad (27)$$

$$\begin{aligned}
 P_{8n} &= R_1 T_{1,n-2} + R_3 S_{3,n-2} + R_5 S_{3,n-2} + R_4 S_{2n} + R_6 S_{3n} + R_6 S_{4,n-2} \\
 P_{9n} &= R_1 T_{1,n+2} - R_2 S_{5,n+2} + R_3 S_{5,n+2} + R_4 S_{2n} + R_5 S_{4n} + R_6 S_{3,n+2} \\
 P_{10n} &= -R_2 S_{5,n+2} + R_4 S_{2,n+2} + R_6 S_{4,n+2} \\
 P_{11n} &= R_2 T_{9,n-2} + R_4 T_{2,n-2} + R_6 T_{3,n-2} \\
 P_{12n} &= R_1 S_{1,n-2} + R_4 T_{3,n-2} + R_3 T_{2,n-2} + R_4 T_{2n} + R_5 T_{3n} + R_6 T_{4,n-2} \\
 P_{13n} &= R_1 S_{1,n+2} - R_2 T_{4,n+2} + R_3 T_{2,n+2} + R_4 T_{2n} + R_5 T_{4n} + R_6 T_{3,n+2} \\
 P_{14n} &= -R_2 T_{4,n+2} + R_4 T_{2,n+2} + R_6 T_{4,n+2}
 \end{aligned}$$

The constants U_{mn} are analogous to the quantities P_{mn} with the R_n in each case replaced by W_n , the S_{mn} by X_{mn} , and the T_{mn} by Y_{mn} . The quantities R_n are defined as follows:

$$\begin{aligned}
 R_1 &= -(1-\nu) \sinh 2\xi_1 & R_4 &= -\frac{1}{2} \\
 R_2 &= \frac{1}{2}(1-\nu) & R_5 &= -\nu \cosh 2\xi_1 \\
 R_3 &= \cosh 2\xi_1 & R_6 &= \frac{1}{2}\nu
 \end{aligned} \tag{28}$$

The quantities W_n are analogous to the R_n with ξ_1 replaced by ξ_2 . The constants S_{mn} , T_{mn} , X_{mn} , and Y_{mn} are given by the relations:

$$\begin{aligned}
 S_{1n} &= n \sinh n\xi_1 & S_{6n} &= n^2(n+2) \sinh (n+2)\xi_1 \\
 S_{2n} &= n^2 \sinh n\xi_1 & S_{7n} &= n^2(n-2) \sinh (n-2)\xi_1 \\
 S_{3n} &= n^2 \sinh (n-2)\xi_1 & S_{8n} &= n \sinh (n+2)\xi_1 \\
 S_{4n} &= n^2 \sinh (n+2)\xi_1 & S_{9n} &= n \sinh (n-2)\xi_1 \\
 S_{5n} &= n^3 \sinh n\xi_1 & &
 \end{aligned} \tag{29}$$

The constants T_{mn} are analogous to the S_{mn} with the hyperbolic cosine function replacing the hyperbolic sine in every instance. The quantities X_{mn} correspond to the S_{mn} with ξ_1 replaced by ξ_2 . The same replacement in the expressions for T_{mn} gives the constants Y_{mn} .

$$\begin{aligned}
 \text{Also: } S'_{1n} &= \sinh n\xi_1 & X'_{1n} &= \sinh n\xi_2 \\
 T'_{1n} &= \cosh n\xi_1 & Y'_{1n} &= \cosh n\xi_2
 \end{aligned} \tag{30}$$

The infinite set set of unknowns B_n , D_n , F_n , and H_n may be evaluated from Equations (24). The set of equations analogous to these but containing the unknowns A_n , C_n , E_n , and G_n may be solved for these unknowns. Thus, all of the coefficients of the terms in Equation (15) may be found by solving successive groups of simultaneous linear algebraic equations.

The series given by Equation (15) converges absolutely and uniformly in both variables throughout the plate, including the boundary.

The Second Problem

Statement of the Problem. The boundary condition (a) of the first problem is replaced by the new condition:

(a') The inner edge of the plate, $\xi = \xi_1$, is not supported against vertical deflection.

The remaining three boundary conditions are as stated in the first problem.

In rectangular coordinates, the condition of no support at an edge ($x = a$) is expressed by the equation

$$\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu) \frac{\partial^3 w}{\partial x \partial y^2} \right]_{x=a} = 0 \tag{31}$$

This expression may be transformed to elliptic coordinates by use of Equations (3). It becomes

$$\begin{aligned}
 & [3(3-\nu) \sinh^3 \xi_1 + 2(\nu-2) \sinh^3 \xi_1 \cosh \xi_1 + 2 \sinh^5 \xi_1] \left[\frac{\partial w}{\partial \xi} \right]_{\xi=\xi_1} \\
 & - (1+\nu) [\sinh^4 \xi_2 \cosh \xi_1] \left[\frac{\partial^2 w}{\partial \xi^2} \right]_{\xi=\xi_1} + [\sinh^5 \xi_1] \left[\frac{\partial^3 w}{\partial \xi^3} \right]_{\xi=\xi_1} \\
 & + [\sinh^5 \xi_1] \left[\frac{\partial^3 w}{\partial \xi \partial \eta^2} \right]_{\xi=\xi_1} - 2 [\sinh^4 \xi_1 \cosh \xi_1] \left[\frac{\partial^3 w}{\partial \eta^3} \right]_{\xi=\xi_1}
 \end{aligned} \tag{32}$$

The combination of harmonic and biharmonic terms used in the first problem may again be assumed to represent the middle surface of the plate. This is given by Equation (15).

Let us introduce the additional notation :

$$\begin{aligned} Q_{1n} &= R_7 T_{1n} - R_8 S_{2n} + R_{10} S_{3n} \\ Q_{2n} &= R_7 S_{1n} - R_8 T_{2n} + R_{10} T_{3n} \\ Q_{3n} &= R_7 T_{1,n+2} - R_8 S_{2,n+2} + R_9 T_{5,n+2} - R_9 T_{6n} + R_{10} S_{4n} \\ Q_{4n} &= R_7 T_{1,n-2} - R_8 S_{2,n-2} + R_9 T_{5,n-2} - R_9 T_{7n} + R_{10} S_{3n} \\ Q_{5n} &= R_7 S_{1,n+2} - R_8 T_{2,n+2} + R_9 S_{3,n+2} - R_9 S_{6n} + R_{10} T_{4n} \\ Q_{6n} &= R_7 S_{1,n-2} - R_8 T_{2,n-2} + R_9 S_{3,n-2} - R_9 S_{7n} + R_{10} T_{3n} \end{aligned} \quad (38)$$

where

$$\begin{aligned} R_7 &= (9-3\nu) \sinh^3 \xi_1 - (2\nu-4) \sinh^3 \xi_1 \cosh^2 \xi_1 - 2 \sinh^5 \xi_1 \\ R_8 &= (1+\nu) \sinh^4 \xi_1 \cosh \xi_1 \\ R_9 &= \sinh^5 \xi_1 \\ R_{10} &= 2 \sinh^4 \xi_1 \cosh \xi_1 \end{aligned} \quad (34)$$

The boundary condition (a'), as expressed by Equation (32) may now be applied to the expression for w given by Equation (15). The following equations are obtained by equating to zero the coefficients of the various sine and cosine terms :

$$\begin{aligned} E_0 Q_{02} + H_0 Q_{03} &= 0 \\ F_1 Q_{31} + H_1 Q_{51} &= 0 \\ B_2 Q_{12} + D_2 Q_{22} + F_2 Q_{32} + H_2 Q_{52} + 4H_0 &= 0 \\ B_n Q_{1n} + D_n Q_{2n} + F_n Q_{3n} + F_{n-2} Q_{4n} + H_n Q_{5n} + H_{n-2} Q_{6n} &= 0 \quad n = 3, 4, \dots \\ E_1 Q_{31} + G_1 Q_{51} &= 0 \\ E_2 Q_{32} + G_2 Q_{52} &= 0 \\ A_n Q_{1n} + C_n Q_{2n} + E_n Q_{3n} + E_{n-2} Q_{4n} + G_n Q_{5n} + G_{n-2} Q_{6n} &= 0 \quad n = 3, 4, \dots \end{aligned} \quad (35)$$

These seven equations, in the order given, replace the first equation in each of the groups (21), (22), etc., respectively. The other equations in each of these groups remain unaltered. Thus, we again have one infinite set of equations for the infinite set of unknowns B_n , D_n , F_n and H_n ; and another infinite set of equations for the infinite set of unknowns A_n , C_n , E_n , and G_n . The coefficients of the terms appearing in Equation (15) may be obtained by solving these linear equation.

The Third Problem

Statement of the Problem. The boundary condition (b) of the first problem is replaced by the new condition :

(b') The outer edge of the plate, $\xi = \xi_2$, is not supported against vertical deflection.

The remaining three boundary conditions are stated in the first problem.

The condition prevailing at the outer edge may be expressed by an equation analogous to Equation (32), but with ξ_1 replaced by ξ_2 .

Expressions similar to those given in Equations (33) and (34) may be introduced with ξ_2 replacing ξ_1 , and equations analogous to Equations (35) and (36) will be obtained.

These equations will replace the second equation in each of the groups (21), (22), etc. respectively. The other equations in each of these groups remain unaltered. Again, the coefficients of the terms appearing in Equation (15) be obtained by solving these linear equations.

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SOME NEW PROPERTIES OF GENERALISED LAPLACE TRANSFORM

By

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1. The generalisation of the Laplace integral, given by Dr. R. S. Varma and according to him known as the Whittaker transform, is

$$\varphi(p) = p \int_0^{\infty} (2xp)^{-\frac{1}{2}} W_{k,m}(2px) h(x) dx \quad (1)$$

to which I have given the symbolic notation (Bose, 1949)

$$\varphi(p) \stackrel{k}{\underset{m}{=}} h(x).$$

For $k = \frac{1}{2}$, $m = \pm \frac{1}{2}$, the above transform reduces to the Laplace transform, since we have

$$(2xp)^{-\frac{1}{2}} W_{\frac{1}{2}, \pm \frac{1}{2}}(2xp) \equiv e^{-px}$$

Other particular cases of this transform are:

(i) When $k = 0$, we have

$$W_{0,m}(x) = (x/\pi)^{\frac{1}{2}} K_m(x/2),$$

the general transform reduces to

$$\varphi(p) = (p/\sqrt{\pi}) \int_0^{\infty} (2xp)^{\frac{1}{2}} K_m(px) h(x) dx$$

and is known as K -transform and symbolically denoted by

$$\varphi(p) \stackrel{0}{\underset{m}{=}} h(x).$$

(ii) When $k = \frac{1}{2} + \frac{1}{2}n + l$ and $m = \pm \frac{1}{2}n$, we have

$$W_{\frac{1}{2} + \frac{1}{2}n + l, \pm \frac{1}{2}n}(x) = (-)^l e^{-\frac{1}{2}x} x^{\frac{1}{2}n + \frac{1}{2}} L_l^n(x),$$

l being a positive integer and the general transform gives

$$\varphi(p) = (-)^l p \int_0^{\infty} (2xp)^{\frac{1}{2}n + \frac{1}{2}} e^{-px} L_l^n(2xp) h(x) dx$$

and known as L_l^n -transform and symbolically denoted by

$$\varphi(p) \stackrel{\frac{1}{2} + \frac{1}{2}n + l}{\underset{\pm \frac{1}{2}n}{=}} h(x)$$

(iii) When $k = \frac{1}{2}n + \frac{1}{2}$, $m = \pm \frac{1}{2}$, we have

$$D_n(2\sqrt{x}) = 2^{1/n}(2x)^{-1/4} W_{\frac{1}{2}n+\frac{1}{2}, \frac{1}{2}}(2x)$$

and the general transform reduces to

$$\varphi(p) = 2^{-1/n} p \int_0^\infty D_n\{2(xp)^{1/2}\} h(x) dx$$

and is known as D_n -transform and symbolically denoted by

$$\varphi(p) \stackrel{\frac{1}{2}n+\frac{1}{2}}{\underset{\pm\frac{1}{2}}{=}} h(x)$$

In this paper I have given some new properties of this new transform, the analogues of which do not exist in the case of Laplace transform. Properties of Whittaker function, Parabolic Cylinder function and others have been utilized to find these new properties. The results of this paper are interesting and general,

2. Theorem I. *The recurrence formula that holds for $W_{k,m}(z)$ also holds for the Whittaker transform of the function $x^{-\lambda}f(x)$, where λ is any arbitrary parameter, provided the integrals and the series involved are convergent.*

Proof: (a) We know

$$W_{k,m}(z) = z^{1/2} W_{k-\frac{1}{2}, m-\frac{1}{2}}(z) + (\frac{1}{2}-k+m) W_{k-1, m}(z). \quad (A)$$

Let

$$\varphi_{k,m,\lambda}(p) \stackrel{k}{\underset{m}{=}} x^{-\lambda} f(x)$$

that is

$$\varphi_{k,m,\lambda}(p) = p \int_0^\infty (2xp)^{-1/2} W_{k,m}(2xp) x^{-\lambda} f(x) dx \quad (1)$$

$$\begin{aligned} &= p \int_0^\infty x^{-\lambda} (2xp)^{-1/2} f(x) [(2xp)^{1/2} W_{k-\frac{1}{2}, m-\frac{1}{2}}(2xp) + (\frac{1}{2}-k+m) W_{k-1, m}(2xp)] dx \\ &= (2p)^{1/2} \varphi_{k-\frac{1}{2}, m-\frac{1}{2}, \lambda-\frac{1}{2}}(p) + (\frac{1}{2}-k+m) \varphi_{k-1, m, \lambda}(p). \end{aligned}$$

Hence, we get the recurrence formula

$$\varphi_{k,m,\lambda}(p) = (2p)^{1/2} \varphi_{k-\frac{1}{2}, m-\frac{1}{2}, \lambda-\frac{1}{2}}(p) + (\frac{1}{2}-k+m) \varphi_{k-1, m, \lambda}(p) \quad (2)$$

provided the integrals in (1) are uniformly convergent.

Integrals in (1) are uniformly and absolutely convergent, if $\mathbf{R}(\mu-\lambda \pm m + \frac{3}{4}) > 0$ where $f(x) = O(x^\mu)$ for small x and $(2xp_0)^{-1/2} W_{k-1, m}(2xp_0) x^{-\lambda} f(x)$ is bounded for $x \geq 0$ and $\mathbf{R}(p) > \mathbf{R}(p_0) > 0$.

(b) We know

$$W_{k,m}(z) = z^{1/2} W_{k+\frac{1}{2}, m+\frac{1}{2}}(z) + (\frac{1}{2}-k-m) W_{k-1, m}(z). \quad (B)$$

Hence, as in (a), we can easily obtain a recurrence formula

$$\varphi_{k,m,\lambda}(p) = (2p)^{1/2} \varphi_{k+\frac{1}{2}, m+\frac{1}{2}, \lambda+\frac{1}{2}}(p) + (\frac{1}{2}-k-m) \varphi_{k-1, m, \lambda}(p) \quad (3)$$

provided $\mathbf{R}(\mu-\lambda \pm m + \frac{3}{4}) > 0$ where $f(x) = O(x^\mu)$ for small values of x and $\mathbf{R}(p) > \mathbf{R}(p_0) > 0$.

3. Here we point out certain interesting results.

(i) If in (2), we put $k = \frac{1}{2}n + \frac{1}{2}$, $m = \pm \frac{1}{2}$, then the left hand side reduces to D_n -transform where as the right hand side remains W -transform. Thus it is interesting to note that the sum of two Whittaker transforms is equivalent to a D_n -transform for any function $x^{-\lambda}f(x)$, for which the integrals and the series involved are convergent.

(ii) If in (3), we put $k = \frac{1}{2}n + \frac{1}{2}$, $m = \pm \frac{1}{2}$, again we mark the same result as in (i).

(iii) If in (2) and (3), we put $k = 0$, we find that the left hand side reduces to K -transform where as the right hand side is the sum of two W -transforms. Hence in this case also we mark the same result.

(iv) If in (2) and (3), we put $k = \frac{1}{2} + \frac{1}{2}n + l$ and $m = \pm \frac{1}{2}n$, then again we observe the same result as in (i) for L_l^n -transform.

4. *Example.* Let (Bose, 1949)

$$x^{-\lambda}f(x) = x^n e^{-qx} \frac{k}{m} \frac{\Gamma(n+m+\frac{5}{4})\Gamma(n-m+\frac{5}{4})}{2(2p)^n \Gamma(n-k+\frac{7}{4})} {}_2F_1 \left\{ \begin{matrix} n+m+\frac{7}{4}, n-m+\frac{5}{4} \\ n-k+\frac{7}{4} \end{matrix} ; \frac{1}{2} - \frac{q}{2p} \right\},$$

$$R(n \pm m + \frac{5}{4}) > 0, R(p) > R(p_0) > 0 \text{ and } |p| > |q|,$$

then from (2.2), we get

$$\begin{aligned} & (n-k+\frac{7}{4}) {}_2F_1 \left\{ \begin{matrix} n+m+\frac{5}{4}, n-m+\frac{5}{4} \\ n-k+\frac{7}{4} \end{matrix} ; \frac{1}{2} - \frac{q}{2p} \right\} \\ &= (n-m+\frac{5}{4}) {}_2F_1 \left\{ \begin{matrix} n+m+\frac{5}{4}, n-m+\frac{5}{4} \\ n-k+\frac{11}{4} \end{matrix} ; \frac{1}{2} - \frac{q}{2p} \right\} \\ &+ (\frac{1}{2}-k+m) {}_2F_1 \left\{ \begin{matrix} n+m+\frac{5}{4}, n-m+\frac{5}{4} \\ n-k+\frac{11}{4} \end{matrix} ; \frac{1}{2} - \frac{q}{2p} \right\} \\ &R(n \pm m + \frac{5}{4}) > 0, R(p) > R(p_0) > 0 \text{ and } |p| > |q|. \end{aligned}$$

This may be written in the form

$$\begin{aligned} (\gamma - \frac{7}{4}) {}_2F_1 \left\{ \begin{matrix} \alpha + \frac{5}{4}, \beta + \frac{5}{4} \\ \gamma + \frac{7}{4} \end{matrix} ; x \right\} &= (\beta + \frac{5}{4}) {}_2F_1 \left\{ \begin{matrix} \alpha + \frac{5}{4}, \beta + \frac{5}{4} \\ \gamma + \frac{11}{4} \end{matrix} ; x \right\} \\ &+ (\frac{1}{2} + \gamma - \beta) {}_2F_1 \left\{ \begin{matrix} \alpha + \frac{5}{4}, \beta + \frac{5}{4} \\ \gamma + \frac{11}{4} \end{matrix} ; x \right\}. \end{aligned}$$

5. **Theorem II.** If the Whittaker transform of $x^{-\lambda}f(x)$ with respect to the first derivative of Whittaker function be $\Phi'_{k,m,\lambda}(p)$, that is

$$\Phi'_{k,m,\lambda}(p) \stackrel{k}{=} x^{-\lambda}f(x) \quad (1)$$

then

$$2p\Phi'_{k,m,\lambda}(p) = k\varphi_{k,m,\lambda+1}(p) - p\varphi_{k,m,\lambda}(p) - \{m^2 - (k - \frac{1}{2})^2\}\varphi_{k-1,m,\lambda+1}(p).$$

Proof: We know

$$zW'_{k,m}(z) = (k - \frac{1}{2})W_{k,m}(z) - \{m^2 - (k - \frac{1}{2})^2\}W_{k-1,m}(z) \quad (c)$$

From (1), we have

$$\begin{aligned}\Phi'_{k,m,\lambda}(p) &= p \int_0^\infty (2xp)^{-\frac{1}{2}} W'_{k,m}(2xp) x^{-\lambda} f(x) dx \\ &= \frac{p}{2p} \int_0^\infty (2xp)^{-\frac{1}{2}} x^{-\lambda-1} f(x) [(k-px)W_{k,m}(2px) - \{m^2 - (k-\frac{1}{2})^2\}W_{k-1,m}(2px)] dx. \quad (2)\end{aligned}$$

Hence

$$2p\Phi'_{k,m,\lambda}(p) = k\varphi_{k,m,\lambda+1}(p) - p\varphi_{k,m,\lambda}(p) - \{m^2 - (k-\frac{1}{2})^2\}\varphi_{k-1,m,\lambda+1}(p) \quad (3)$$

provided $\mathbf{R}(\mu - \lambda \pm m + \frac{1}{2}) > 0$ where $f(x) = O(x^\mu)$ for small values of x and $(2xp_0)^{-\frac{1}{2}} x^{-\lambda-1} f(x) W_{k,m}(2p_0 x)$ is bounded for $x \geq 0$, $\mathbf{R}(p) > \mathbf{R}(p_0) > 0$

6. *Example.* Let (Bose, 1949)

$$f(x) = x^n \frac{k}{m} \frac{\Gamma(n+m+\frac{5}{4})\Gamma(n-m+\frac{5}{4})}{2^{n+1} p^n \Gamma(n-k+\frac{7}{4})} {}_2F_1 \left\{ \begin{matrix} n+m+\frac{5}{4}, n-m+\frac{5}{4} \\ n-k+\frac{7}{4} \end{matrix} ; \frac{1}{2} \right\}$$

$$\mathbf{R}(n \pm m + \frac{5}{4}) > 0 \text{ and } \mathbf{R}(p) > \mathbf{R}(p_0) > 0,$$

then from (5.3), we get

$$\begin{aligned}\Phi'_{k,m,\lambda}(p) &= \frac{1}{2(2p)^{n-\lambda}} \left[\frac{k\Gamma(n-\lambda+m+\frac{1}{4})\Gamma(n-\lambda-m+\frac{1}{4})}{\Gamma(n-\lambda-k+\frac{3}{4})} \right. \\ &\quad \times {}_2F_1 \left\{ \begin{matrix} n-\lambda+m+\frac{1}{4}, n-\lambda-m+\frac{1}{4} \\ n-\lambda-k+\frac{3}{4} \end{matrix} ; \frac{1}{2} \right\} - \frac{1}{2} \frac{\Gamma(n-\lambda+m+\frac{5}{4})\Gamma(n-\lambda-m+\frac{5}{4})}{\Gamma(n-\lambda-k+\frac{7}{4})} \\ &\quad \times {}_2F_1 \left\{ \begin{matrix} n-\lambda+m+\frac{5}{4}, n-\lambda-m+\frac{5}{4} \\ n-\lambda-k+\frac{7}{4} \end{matrix} ; \frac{1}{2} \right\} - \frac{\{m^2 - (k-\frac{1}{2})^2\}\Gamma(n-\lambda+m+\frac{1}{4})\Gamma(n-\lambda-m+\frac{1}{4})}{\Gamma(n-\lambda-k+\frac{3}{4})} \\ &\quad \left. {}_2F_1 \left\{ \begin{matrix} n-\lambda+m+\frac{1}{4}, n-\lambda-m+\frac{1}{4} \\ n-\lambda-k+\frac{3}{4} \end{matrix} ; \frac{1}{2} \right\} \right] \mathbf{R}(n-\lambda \pm m + \frac{1}{2}) > 0 \text{ and } \mathbf{R}(p) > \mathbf{R}(p_0) > 0.\end{aligned}$$

7. **Theorem III.** If

$$\varphi_1(p) \stackrel{\frac{1}{2}n+\frac{1}{4}}{\pm \frac{1}{4}} f(x)$$

and

$$\psi(p) \doteq x^{\frac{1}{2}n} \varphi(x)$$

then

$$\psi(p) = \frac{1}{(2p)^{\frac{1}{2}n}} \left[\varphi_m(p) + \frac{m(m-1)}{2^2 1!} \varphi_{m-2}(p) + \frac{m(m-1)(m-2)(m-3)}{2^4 2!} \varphi_{m-4}(p) + \dots \right]$$

where the last term being

$$\frac{2^{3/2} m!}{(8p)^{\frac{1}{2}m} \{\frac{1}{2}(m-1)\}!} \varphi_1(p), \quad \text{when } m \text{ is odd}$$

and

$$\frac{m!}{(8p)^{\frac{1}{2}m} (\frac{1}{2}m)!} \varphi_0(p) \quad \text{when } m \text{ is even.}$$

Proof: We know (Varma, 1938)

$$z^m e^{-\frac{1}{2}z} = D_m(z) + \frac{m(m-1)}{2 \cdot 1!} D_{m-2}(z) + \frac{m(m-1)(m-2)(m-3)}{2^2 \cdot 2!} D_{m-4}(z) + \dots$$

the last term being

$$\frac{m!}{2^{\frac{1}{2}m} \left\{ \frac{1}{2}(m-1) \right\}!} D_1(z), \quad \text{when } m \text{ is odd}$$

$$\frac{m!}{2^{\frac{1}{2}m} \left(\frac{1}{2}m \right)!} D_0(z), \quad \text{when } m \text{ is even.}$$

Also, we have

$$2^{\frac{1}{2}r} \varphi_r(p) = p \int_0^\infty D_r \{2(xp)^{\frac{1}{2}}\} f(x) dx \quad (2)$$

and, if

$$\varphi_r(p) \stackrel{k}{=} h_r(x), \text{ then}$$

$$\sum_{r=0}^n \varphi_r(p) \stackrel{k}{=} \sum_{r=1}^n h_r(x).$$

Using these results, we get

$$\frac{1}{(2p)^{\frac{1}{2}m}} \left[\varphi_m(p) + \frac{m(m-1)}{2 \cdot 1!} \varphi_{m-2}(p) + \frac{m(m-1)(m-2)(m-3)}{2^2 \cdot 2!} \varphi_{m-4}(p) + \dots \right] = \psi(p)$$

where $\psi(p) \doteq x^{\frac{1}{2}m} f(x)$ and the last term

$$\frac{1}{(8p)^{\frac{1}{2}m}} \cdot \frac{2^{3/2}m!}{\left\{ \frac{1}{2}(m-1) \right\}!} \varphi_1(p), \quad \text{when } m \text{ is odd}$$

and

$$\frac{1}{(8p)^{\frac{1}{2}m}} \cdot \frac{m!}{\left(\frac{1}{2}m \right)!} \varphi_0(p), \quad \text{when } m \text{ is even}$$

provided $\mathbf{R}(\mu) > -1$ where, $f(x) = O(x^\mu)$ for small values of x and $\mathbf{R}(p) > \mathbf{R}(p_0) > 0$.

8. Theorem III helps us in simplifying big expressions which, in general, we come across when we find the D_n -transforms. Generally, we find that Laplace transforms are simple expressions and so the interesting thing that we mark in Theorem III is that the sum of a number of D_n -transforms is a Laplace transform which, in general, is a simple expression.

This can be illustrated by the following examples.

(i) Let (Bose, 1949)

$$f(x) = x^n e^{-qx} \stackrel{\frac{1}{2}r + \frac{1}{2}}{=} \frac{\Gamma(n+1)\Gamma(n+\frac{3}{2})}{\pm \frac{1}{2} 2(2p)^n \Gamma(n-\frac{1}{2}r+\frac{3}{2})} \cdot {}_2F_1 \left\{ \begin{matrix} n+1, n+\frac{3}{2} \\ n-\frac{1}{2}r+\frac{3}{2} \end{matrix} ; \frac{1}{2} - \frac{q}{2p} \right\}$$

$$\mathbf{R}(n) > -1, \mathbf{R}(p) > \mathbf{R}(p_0) > 0 \text{ and } |p| > |q|$$

$$\psi(p) = \frac{\Gamma(n+\frac{1}{2}m+1)p}{(p+q)^{n+\frac{1}{2}m+1}} \doteq x^{m/2+n} e^{-qx}, \mathbf{R}(n+m/2+1) > 0.$$

Hence by Theorem III, we get

$$\frac{\Gamma(n + \frac{1}{2}m + 1)p}{(p+q)^{n+\frac{1}{2}m+1}} = \frac{\Gamma(n+1)\Gamma(n+\frac{3}{2})}{2(2p)^{m/2+n}} \left[\frac{1}{\Gamma(n-\frac{1}{2}m+\frac{3}{2})} {}_2F_1 \left\{ \begin{matrix} n+1, n+\frac{3}{2} \\ n-\frac{1}{2}m+\frac{3}{2} \end{matrix} ; \frac{1}{2} - \frac{q}{2p} \right\} \right. \\ \left. + \frac{m(m-1)}{2^2 1! \Gamma(n-\frac{1}{2}m+\frac{5}{2})} {}_2F_1 \left\{ \begin{matrix} n+1, n+\frac{5}{2} \\ n-\frac{1}{2}m+\frac{5}{2} \end{matrix} ; \frac{1}{2} - \frac{q}{2p} \right\} + \dots \right]$$

and the last term is

$$\frac{m! \Gamma(n+\frac{3}{2})}{2^{m-1} (2p)^{\frac{1}{2}m-8/2} \{\frac{1}{2}(m-1)\}! (p+q)^{\frac{1}{2}n+8/2}}, \quad \text{when } m \text{ is odd}$$

and

$$\frac{m! \Gamma(n+1)}{2^{m+1} (2p)^{\frac{1}{2}m-1} (\frac{1}{2}m)! (p+q)^{n+1}}, \quad \text{when } m \text{ is even.}$$

(ii) Let (Bose, 1949)

$$f(x) = x^\mu J_n(bx) \stackrel{\frac{1}{2}r+\frac{1}{2}}{\pm \frac{1}{2}} \frac{(b/2)^n}{2(2p)^{n+\mu}} \sum_{s=0}^{\infty} \frac{\Gamma(n+\mu+2s+\frac{3}{2})\Gamma(n+\mu+2s+1)}{s! \Gamma(n+s+1)\Gamma(n+\mu+2s-\frac{1}{2}r+\frac{3}{2})} \\ \times \left(\frac{-b^2}{16p^2} \right)^s {}_2F_1 \left\{ \begin{matrix} n+\mu+2s+\frac{3}{2}, n+\mu+2s+1 \\ n+\mu-\frac{1}{2}r+2s+\frac{3}{2} \end{matrix} ; \frac{1}{2} \right\},$$

$$R(n+\mu+1) > 0, R(p) > R(p_0) > 0 \text{ and } |p| > |b|,$$

$$\psi(p) = \frac{(b/2)^n \Gamma(n+\mu+\frac{1}{2}m+1)}{\Gamma(n+1)p^{n+\mu+\frac{1}{2}m}} \left\{ \begin{matrix} \frac{1}{2}\mu+\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}, \frac{1}{2}\mu+\frac{1}{2}m+\frac{1}{2}n+1 \\ n+1 \end{matrix} ; -b^2/p^2 \right\}.$$

Hence by Theorem III, we get

$$\frac{\Gamma(n+\mu+\frac{1}{2}m+1)}{\Gamma(n+1)} {}_2F_1 \left\{ \begin{matrix} \frac{1}{2}\mu+\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}, \frac{1}{2}\mu+\frac{1}{2}m+\frac{1}{2}n+1 \\ n+1 \end{matrix} ; -b^2/p^2 \right\} \\ = \frac{1}{2^{n+\mu+\frac{1}{2}m+1}} \left[\sum_{s=0}^{\infty} \frac{\Gamma(n+\mu+2s+\frac{3}{2})\Gamma(n+\mu+2s+1)}{s! \Gamma(n+s+1)\Gamma(n+\mu-\frac{1}{2}m+2s+\frac{3}{2})} \left(-\frac{b^2}{16p^2} \right)^s \right. \\ \times {}_2F_1 \left\{ \begin{matrix} n+\mu+2s+\frac{3}{2}, n+\mu+2s+1 \\ n+\mu-\frac{1}{2}m+2s+\frac{5}{2} \end{matrix} ; \frac{1}{2} \right\} + \frac{m(m-1)}{2^2 1!} \\ \times \sum_{s=0}^{\infty} \frac{\Gamma(n+\mu+2s+\frac{3}{2})\Gamma(n+\mu+2s+1)(-b^2/16p^2)^s}{s! \Gamma(n+s+1)\Gamma(n+\mu+2s-\frac{1}{2}m+\frac{5}{2})} \\ \left. \times {}_2F_1 \left\{ \begin{matrix} n+\mu+2s+\frac{3}{2}, n+\mu+2s+1 \\ n+\mu-\frac{1}{2}m+2s+\frac{5}{2} \end{matrix} ; \frac{1}{2} \right\} + \dots \right].$$

9. Theorem IV. If

$$\varphi_n(p) \stackrel{\frac{1}{2}+\frac{1}{2}\alpha+n}{\pm \frac{1}{2}\alpha} f(x)$$

and

$$\varphi(p) \doteq (x/a)^{\frac{1}{2}\alpha+\frac{1}{2}} f(x/a), \quad \text{where } a = \frac{1+t}{1-t}$$

then

$$\varphi(p) = \frac{(1-t)^{\alpha}(1+t)}{(2p)^{\frac{1}{2}\alpha+\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(-)^n t^n}{n!} \varphi_n(p)$$

provided $\mathbf{R}(\mu + \frac{1}{2}\alpha + \frac{1}{2}) > 0$, where $f(x) = O(x^{\mu})$ for small values of x , $\mathbf{R}(p) > \mathbf{R}(p_0) > 0$, $|t| < 1$ and $f(x)$ is continuous.

Proof: We have

$$\varphi_n(p) = (-)^n n! p \int_0^{\infty} (2xp)^{\frac{1}{2}\alpha+\frac{1}{2}} e^{-xp} L_n^{\alpha}(2xp) f(x) dx \quad (1)$$

Multiplying by $(-)^n t^n / n!$ and taking the sum from $n = 0$ to $n = \infty$, we get (Howel, 1937)

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-)^n t^n}{n!} \varphi_n(p) &= p \int_0^{\infty} (2xp)^{\frac{1}{2}\alpha+\frac{1}{2}} e^{-xp} \frac{e^{-2pxt/(1-t)}}{(1-t)^{\alpha+1}} f(x) dx, \\ &= \frac{(2p)^{\frac{1}{2}\alpha+\frac{1}{2}}}{(1+t)(1-t)^{\alpha}} \varphi(p), \quad |t| < 1 \end{aligned}$$

where

$$\varphi(p) = (x/a)^{\frac{1}{2}\alpha+\frac{1}{2}} f(x/a), \quad a = \frac{1-t}{1+t}, \quad \mathbf{R}(p) > \mathbf{R}(p_0) > 0, \quad |t| < 1 \text{ and } \mathbf{R}(\mu + \frac{1}{2}\alpha + \frac{1}{2}) > 0.$$

Hence

$$\varphi(p) = \frac{(1-t)^{\alpha}(1+t)}{(2p)^{\frac{1}{2}\alpha+\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(-)^n t^n}{n!} \varphi_n(p). \quad (2)$$

The change of order of integration and summation is justified due to the uniform and absolute convergence of the series and the integral for $\mathbf{R}(p) > \mathbf{R}(p_0) > 0$ and $\mathbf{R}(\mu + \frac{1}{2}\alpha + \frac{1}{2}) > 0$, where $f(x) = O(x^{\mu})$ for small x .

10. Theorem V. If

$\Phi_{n-r}^{\alpha+2r}(p)$ is the $L_{n-r}^{\alpha+2r}$ -transform of $x^r f(x)$,

and

$$\psi(p) = \frac{\frac{1}{2} + \alpha + \frac{1}{2} + 2n}{\pm(\alpha + \frac{1}{2})} x^{-\frac{1}{2}\alpha-\frac{1}{2}} \varphi(x),$$

then

$$\psi(p) = \frac{(2p)^{\frac{1}{2}\alpha+\frac{1}{2}} \Gamma(2n+2\alpha+2)}{\Gamma(n+\alpha+1)} \sum_{r=0}^n \frac{(-)^{n-r} \Gamma(\alpha+r+1) (2p)^r}{r! (n-r)! \Gamma(2\alpha+2r+2)} \Phi_{n-r}^{\alpha+2r}(p).$$

Proof: We have

$$(2p)^r \Phi_{n-r}^{\alpha+2r}(p) = (-)^{n-r} p(n-r)! \int_0^{\infty} (2xp)^{\frac{1}{2}\alpha+\frac{1}{2}} e^{-xp} \times L_{n-r}^{\alpha+2r}(2xp) \cdot (2xp)^r f(x) dx \quad (1)$$

or

$$\frac{(-)^{n-r}}{(n-r)!} (2p)^r \Phi_{n-r}^{\alpha+2r}(p) = p \int_0^{\infty} (2xp)^{\frac{1}{2}\alpha+\frac{1}{2}} e^{-xp} (2xp)^{2r} L_{n-r}^{\alpha+2r}(2xp) f(x) dx.$$

Multiplying by $\frac{\Gamma(\alpha+r+1)}{r! \Gamma(2\alpha+2r+2)}$ and taking the sum from $r = 0$ to $r = n$, we get (Howell,

1937, p. 399).

$$\sum_{r=0}^n \frac{(-)^{n+r} \Gamma(\alpha+r+1) (2p)^r}{(n-r)! r! \Gamma(2\alpha+2r+2)} \Phi_{n-r}^{\alpha+2r}(p) \\ = p \int_0^\infty (2xp)^{\frac{1}{2}\alpha+\frac{1}{2}} e^{-xp} \frac{(2n)! \Gamma(n+\alpha+1)}{n! \Gamma(2n+2\alpha+2)} L_{2n}^{2\alpha+1}(2px) f(x) dx.$$

Hence

$$\psi(p) = \frac{(2p)^{\frac{1}{2}\alpha+\frac{1}{2}} n! \Gamma(2n+2\alpha+2)}{\Gamma(n+\alpha+1)} \sum_{r=0}^n \frac{(-)^{n-r} \Gamma(\alpha+r+1) (2p)^r}{(n-r)! r! \Gamma(2\alpha+2r+2)} \Phi_{n-r}^{\alpha+2r}(p)$$

where

$$\psi(p) x^{\frac{1}{2}\alpha+\frac{1}{2}+2n} x^{-\frac{1}{2}\alpha-\frac{1}{2}} f(x).$$

11. Theorem VI. If

$\varphi_{n-r}(p)$ is L_{n-r}^β transform of $f(x)$, and $\psi(p)$ is L_n^α transform of $x^{\frac{1}{2}(\beta-\alpha)} f(x)$, then

$$\psi(p) = \frac{n!}{\Gamma(\alpha-\beta) (2p)^{\frac{1}{2}(\beta-\alpha)}} \sum_{r=0}^n \frac{(-)^r \Gamma(\alpha-\beta+r)}{r! (n-r)!} \varphi_{n-r}(p).$$

Proof: We have

$$\varphi_{n-r}(p) = (-)^{n-r} (n-r)! p \int_0^\infty (2xp)^{\frac{1}{2}\alpha+\frac{1}{2}} e^{-px} L_{n-r}^\beta(2px) f(x) dx \quad (1)$$

Also, we know that (Howel, 1937, p. 397)

$$L_n^\alpha(x) = \sum_{r=0}^n \frac{\Gamma(\alpha-\beta+r)}{r! \Gamma(\alpha-\beta)} L_{n-r}^\beta(x).$$

Using this, we get

$$\sum_{r=0}^n \frac{\Gamma(\alpha-\beta+r) (-)^{n-r}}{\Gamma(\alpha-\beta) r! (n-r)!} \varphi_{n-r}(p) = p \int_0^\infty (2xp)^{\frac{1}{2}\alpha+\frac{1}{2}} e^{-px} L_n^\alpha(2px) f(x) dx = \frac{(2p)^{\frac{1}{2}(\beta-\alpha)}}{(-)^n n!} \psi(p)$$

Hence

$$\frac{n!}{(2p)^{\frac{1}{2}(\beta-\alpha)} \Gamma(\alpha-\beta)} \sum_{r=0}^n \frac{(-)^r \Gamma(\alpha-\beta+r)}{r! (n-r)!} \varphi_{n-r}(p) = \psi(p) \quad (2)$$

where $\psi(p)$ is the L_n^α transform of $x^{\frac{1}{2}(\beta-\alpha)} f(x)$

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ON WHITTAKER TRANSFORM

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1. The integral (Varma 1947)

$$\varphi(p) = p \int_0^{\infty} (2xp)^{-\frac{1}{2}} W_{k,m}(2px) f(x) dx \quad (1)$$

is the generalisation of the well-known Laplace integral

$$\varphi(p) = p \int_0^{\infty} e^{-px} f(x) dx \quad (2)$$

due to the identity

$$(2px)^{-\frac{1}{2}} W_{\frac{1}{2}, \pm \frac{1}{2}}(2px) \equiv e^{-px}$$

Integral (1) (Bose 1949) is symbolically denoted by

$$\varphi(p) \underset{m}{\overset{k}{=}} f(x)$$

and is known as Whittaker transform.

Two of the particular cases of this transform are

(i) when $k = 0$, integral (1) reduces to

$$\varphi(p) = (p/\pi) \int_0^{\infty} (2xp)^{\frac{1}{2}} K_m(2px) f(x) dx \quad (3)$$

where $K_m(x)$ is Bessel function and is known as K -transform;

(ii) when $k = \frac{1}{2} + \frac{1}{2}n + l$, $m = \pm \frac{1}{2}n$, l being a positive integer, integral (1) reduces to

$$\varphi(p) = (-)^l l! p \int_0^{\infty} (2xp)^{\frac{1}{2}n + \frac{1}{2}} e^{-px} L_l^n(2px) f(x) dx$$

where $L_l^n(x)$ is Generalised Laguerre polynomial, and is known as L_l^n -transform.

In this note I have given a few more properties of this new transform by using the properties of Self-reciprocal functions, Whittaker function and Generalised Laguerre polynomial. The analogues of these results do not exist in the case of Laplace transform, except in Theorem I, which is given in the form of a Corollary.

2. **Theorem I.** If

$$\varphi_n(p) \underset{m}{\overset{k}{=}} x^{-n-1} f(x)$$

and

$$\psi(s) \underset{=}{=} x^{-n-5/4} W_{k,m}(2px) f(x)$$

where $f(x)$ is independent of n , then

$$p^{-1}\phi_{n+m_1}(p) = \frac{(2p)^{-\frac{1}{2}}}{\Gamma(m_1)} \int_0^\infty s^{m_1-2} \psi(s) ds \quad (1)$$

provided $\mathbf{R}(\mu_1 - n \pm m + \frac{1}{2}) > \mathbf{R}(m_1) > 0$, where $f(x) = O(x^{\mu_1})$ for small x ,

$$x^{-n-\frac{1}{2}} W_{k,m}(2p_0 x) e^{-sx} f(x)$$

is bounded for $\mathbf{R}(p) > p_0 > 0$, $s \geq 0$ and the integral (1) converges.

Proof: Since

$$\varphi_n(p) \stackrel{k}{=} x^{-n-1} f(x),$$

therefore,

$$\varphi_{n+m_1}(p) = p \int_0^\infty (2xp)^{-\frac{1}{2}} W_{k,m}(2px) x^{-m_1-n-1} f(x) dx \quad (2)$$

On using

$$x^{-m_1} = \frac{1}{\Gamma(m_1)} \int_0^\infty e^{-sx} s^{m_1-1} ds, \quad \mathbf{R}(m_1) > 0$$

in (2), we have

$$\begin{aligned} \varphi_{n+m_1}(p) &= p \int_0^\infty (2xp)^{-\frac{1}{2}} W_{k,m}(2px) x^{-n-1} f(x) dx \times \frac{1}{\Gamma(m_1)} \int_0^\infty e^{-sx} s^{m_1-1} ds \\ &= \frac{p}{\Gamma(m_1)} \int_0^\infty s^{m_1-1} ds \times \int_0^\infty (2xp)^{-\frac{1}{2}} W_{k,m}(2px) x^{-n-1} e^{-sx} f(x) dx \\ &= \frac{(2p)^{-\frac{1}{2}}}{\Gamma(m_1)} p \int_0^\infty s^{m_1-2} \psi(s) ds \end{aligned} \quad (3)$$

Regarding the justification of the change of order of integration we see that in (3), x -integral is uniformly and absolutely convergent if $\mathbf{R}(\mu_1 - n \pm m + \frac{1}{2}) > 0$ where $f(x) = O(x^{\mu_1})$ for small x and $x^{-n-\frac{1}{2}} W_{k,m}(2p_0 x) f(x) e^{-sx}$ is bounded for $\mathbf{R}(p) > p_0 > 0$, $s \geq 0$, s -integral is absolutely and uniformly convergent for $\mathbf{R}(m_1) > 0$, $x > 0$ and the repeated integral exists due to the convergence of (1).

3. Corollary. If in the above theorem, we put $k = \frac{1}{2}$, $m = \pm \frac{1}{2}$, we get the result of Tewari (1943):

$$\text{If } \varphi_n(p) \stackrel{k}{=} x^{-n-1} f(x),$$

where $f(x)$ is not a function of n and $\mathbf{R}(n) < 0$, then

$$\varphi_{n+m_1}(p) = \frac{p}{\Gamma(m_1)} \int_0^\infty \frac{s^{m_1-1}}{(p+s)} \varphi_n(p+s) ds,$$

provided $\mathbf{R}(p) > 0$, $\mathbf{R}(m_1) > 0$ and the integral converges.

4. Example. If

$$f(x) = x^{3/4} e^{-\frac{1}{2}x},$$

then (Humbert and McLachlan 1941, p. 51)

$$\psi(s) = (-)^{k+n+\frac{1}{2}} (2p)^{n+\frac{1}{2}} \Gamma(k-n+\frac{1}{2}) (s/2p)^{k+n-\frac{1}{2}} (s/2p+1)^{n-k-\frac{1}{2}}$$

and (Bose 1949, p. 13)

$$x^{-m_1-n-\frac{1}{2}}e^{-\frac{1}{2}x} = \frac{\Gamma(1-m_1)\Gamma(1-m_1-2n)}{n 2(2p)^{-m_1-n-\frac{1}{2}}\Gamma(3/4-m_1-n-k)} \cdot {}_2F_1\left\{\begin{matrix} 1-m_1, 1-m_1-2n \\ \frac{3}{2}-m_1-n-k \end{matrix}; \frac{1}{2}-\frac{1}{4p}\right\},$$

$$\mathbf{R}(m_1) < 1, \mathbf{R}(m_1+2n) < 1, \mathbf{R}(p) > 0 \text{ and } |p| > \frac{1}{2}.$$

Hence, on applying Theorem I, we obtain

$$\begin{aligned} & \int_0^\infty s^{m_1+k+n-5/2} \left(\frac{s}{2p} + 1\right)^{n-k-\frac{1}{2}} ds \\ &= \frac{(-)^{k+n+\frac{1}{2}}\Gamma(m_1)\Gamma(1-m_1)\Gamma(1-m_1-2n)(2p)^{m_1+k+n-3/2}}{\Gamma(k-n+\frac{1}{2})\Gamma(\frac{3}{2}-m_1-n-k)} \times {}_2F_1\left\{\begin{matrix} 1-m_1, 1-m_1-2n \\ \frac{3}{2}-m_1-n-k \end{matrix}; \frac{1}{2}-\frac{1}{4p}\right\} \end{aligned}$$

provided $0 < \mathbf{R}(m_1) < 1, \mathbf{R}(m_1+k+n) > \frac{3}{2}, \mathbf{R}(m_1+2n) < 1, \mathbf{R}(p) > 0$, and $|p| > \frac{1}{2}$.

5. Theorem II. If

$$\varphi(p) \stackrel{0}{=} f(x),$$

and $y^{\mu-3/2}\varphi(y)$ is self-reciprocal in the Hankel transform of order ν , then*

$$y^{\mu-\nu-2}\varphi(y) = \frac{2^\mu \Gamma_2(\frac{1}{2}\mu + \frac{1}{2}\nu \pm \frac{1}{2}m + \frac{5}{8})}{\sqrt{(2\pi)} \cdot \Gamma(\nu+1)} \int_0^\infty t^{-\mu-\nu-1} f(t) \cdot {}_2F_1\left\{\begin{matrix} \frac{1}{2}\mu + \frac{1}{2}\nu \pm \frac{1}{2}m + \frac{5}{8} \\ \nu+1 \end{matrix}; -\frac{y^2}{t^2}\right\} dt$$

provided $\mathbf{R}(\nu + \mu \pm m + \frac{5}{4}) > 0, \mathbf{R}(\mu_1 - \mu - \nu) > 0, \mathbf{R}(\mu_1 \pm m + \frac{5}{4}) > 0$ where $f(t) = O(t^{\mu_1})$ for small t , $\mathbf{R}(\mu_1 + \frac{1}{2}\nu + \mu) > 0$ where $\varphi(x) = O(x^{\mu_1})$ for small x , $\mathbf{R}(p) > 0, |p| > |y|$ and $t^{-\mu-\nu}f(t) \cdot {}_2F_1\left\{\begin{matrix} \frac{1}{2}\mu + \frac{1}{2}\nu \pm \frac{1}{2}m + \frac{5}{8} \\ \nu+1 \end{matrix}; -y^2/t^2\right\}$ and $x^{\mu-1}J_\nu(xy)\varphi(x)$ both uniformly tends to zero as $t \rightarrow \infty$ for $y > 0$.

Proof: We have

$$\varphi(p) \stackrel{0}{=} f(x),$$

$$y^{\mu-3/2}\varphi(y) = \int_0^\infty (xy)^{\frac{1}{2}} J_\nu(xy) x^{\mu-3/2} \varphi(x) dx \quad (1)$$

and (Bose 1949, p. 14)

$$\varphi_1(p) = \frac{2^\mu y^\nu \Gamma_2(\frac{1}{2}\mu + \frac{1}{2}\nu \pm \frac{1}{2}m + \frac{5}{8})}{\sqrt{(2\pi)} \cdot \Gamma(\nu+1) p^{\mu+\nu}} \cdot {}_2F_1\left\{\begin{matrix} \frac{1}{2}\mu + \frac{1}{2}\nu \pm \frac{1}{2}m + \frac{5}{8} \\ \nu+1 \end{matrix}; -\frac{y^2}{p^2}\right\}$$

$$\stackrel{0}{=} x^\mu J_\nu(xy) = f_1(x)$$

$$\mathbf{R}(\nu + \mu \pm m + \frac{5}{4}) > 0, \mathbf{R}(p) > 0 \text{ and } |p| > |y|.$$

Now, applying (Bose 1949, p. 19, R. 6): If

$$\varphi(p) \stackrel{0}{=} f(x)$$

* Symbol

$$\Gamma_2(\alpha \pm \beta) = \Gamma(\alpha + \beta) \Gamma(\alpha - \beta)$$

and

$${}_2F_1\left\{\begin{matrix} \alpha \pm \beta \\ \gamma \end{matrix}; x\right\} = {}_2F_1\left\{\begin{matrix} \alpha + \beta, \alpha - \beta \\ \gamma \end{matrix}; x\right\}$$

and

$$\varphi_1(p) \stackrel{0}{=} f_1(x),$$

then

$$\int_0^\infty \varphi_1(t) f(t) \frac{dt}{t} = \int_0^\infty \varphi(x) f_1(x) \frac{dx}{x},$$

we obtain

$$\begin{aligned} & \frac{2^\mu \Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu \pm \frac{1}{2}m + \frac{\delta}{2})}{\sqrt{(2\pi)} \cdot \Gamma(\nu + 1)} \int_0^\infty t^{-\mu-1} f(t) \cdot {}_2F_1 \left\{ \begin{matrix} \frac{1}{2}\mu + \frac{1}{2}\nu \pm \frac{1}{2}m + \frac{\delta}{2} \\ \nu + 1 \end{matrix} ; -\frac{y^2}{t^2} \right\} dt \\ & = y^{-\nu-\frac{1}{2}} \int_0^\infty (xy)^{\frac{1}{2}} J_\nu(xy) x^{\mu-3/2} \varphi(x) dx = y^{\kappa-\nu-2} \varphi(y), \text{ from (1).} \end{aligned}$$

6. Theorem III. *If*

$$\frac{1}{2}r(p) \stackrel{k-n+\tau/2}{=} x^{\frac{1}{2}r} f(x)$$

and

$$\varphi(p) \stackrel{k}{=} f(x),$$

then

$$\varphi(p) = n! \Gamma(m+k+\frac{1}{2}) \sum_{r=0}^n \frac{(-)^{r+n} (2p)^{\frac{1}{2}r}}{(n-r)! r! \Gamma(m+k-n+r+\frac{1}{2})} \varphi_{\frac{1}{2}r}(p)$$

provided $\mathbf{R}(\frac{1}{2}-k+m) > 0$, $\mathbf{R}(\mu \pm m + \frac{\delta}{2}) > 0$ where $f(x) = O(x^\mu)$ for small x , and $x^{3/4+\frac{1}{2}r} W_{\frac{k-n+\tau/2}{m+\frac{1}{2}r}}(2p_0 x) f(x)$ and $x^{3/4} W_{k,m}(2p_0 x) f(x)$ both uniformly tend to zero as $x \rightarrow \infty$ for $\mathbf{R}(p) > p_0 > 0$, $r = 0, 1, \dots, n$.

Proof: We have

$$(2p)^{\frac{1}{2}r} \varphi_{\frac{1}{2}r}(p) = p \int_0^\infty (2xp)^{-\frac{1}{2}} (2px)^{\frac{1}{2}r} W_{\frac{k-n+\frac{1}{2}r}{m+\frac{1}{2}r}}(2px) f(x) dx \quad (1)$$

On multiplying by $\frac{(-)^r}{(n-r)! r! \Gamma(m+k-n+r+\frac{1}{2})}$ and taking the sum from $r = 0$ to $r = n$, and on using (Hari Shanker 1942, p. 52)

$$W_{k,m}(z) = (-)^n \Gamma(m+k+\frac{1}{2}) n! \sum_{r=0}^n \frac{(-)^r z^{\frac{1}{2}r}}{(n-r)! r! \Gamma(m+k-n+r+\frac{1}{2})}$$

$$W_{\frac{k-n+\frac{1}{2}r}{m+\frac{1}{2}r}}(z), \mathbf{R}(\frac{1}{2}-k+m) > 0 \quad (A)$$

we get

$$\sum_{r=0}^n \frac{(-)^r (2p)^{\frac{1}{2}r}}{(n-r)! r! \Gamma(m+k-n+r+\frac{1}{2})} \varphi_{\frac{1}{2}r}(p) = \frac{\varphi(p)}{(-)^n \Gamma(m+k+\frac{1}{2}) n!}$$

provided $\mathbf{R}(\frac{1}{2}-k+m) > 0$, $\mathbf{R}(\mu \pm m + \frac{\delta}{2}) > 0$ where $f(x) = O(x^\mu)$ for small x , and $x^{\frac{1}{2}} W_{k,m}(2p_0 x) f(x)$ and $x^{\frac{1}{2}+\frac{1}{2}r} W_{\frac{k-n+\frac{1}{2}r}{m+\frac{1}{2}r}}(2p_0 x) f(x)$ both uniformly tends to zero as $x \rightarrow \infty$ for $\mathbf{R}(p) > p_0 > 0$, $r = 1, 2, \dots, n$.

7. *Example.* If

$$f(x) = I_\nu(2ax),$$

then (Bose 1949)*, with $k = 0$,

$$\varphi(p) = \frac{(2a)^\nu \Gamma(\frac{1}{2}\nu \pm \frac{1}{2}m + \frac{5}{8})}{\sqrt{(2\pi)} \cdot \Gamma(\nu+1)p^\nu} \cdot {}_2F_1\left\{\begin{matrix} \frac{1}{2}\nu \pm \frac{1}{2}m + \frac{5}{8} \\ \nu+1 \end{matrix}; \frac{4a^2}{p^2}\right\}, \quad \mathbf{R}(\nu \pm m + \frac{5}{4}) > 0, \quad \mathbf{R}(p-2a) > 0$$

and $p | > 2a$,

and

$$\varphi_{\frac{1}{2}r}(p) = \frac{1}{2a^{\frac{1}{2}r}} \sum_{s=0}^{\infty} \frac{\Gamma(\nu+m+r+2s+\frac{5}{2})\Gamma(\nu-m+2s+\frac{5}{4})}{s! \Gamma(\nu+s+1)\Gamma(\nu+n+2s+\frac{7}{4})} \left(\frac{a}{2p}\right)^{\nu+\frac{1}{2}r+2s} \\ \times {}_2F_1\left\{\begin{matrix} \nu+m+r+2s+\frac{5}{4}, \nu-m+2s+\frac{5}{4} \\ \nu+n+2s+\frac{7}{4} \end{matrix}; \frac{1}{2}\right\}$$

$$\mathbf{R}(\nu \pm m + \frac{5}{4}) > 0, \mathbf{R}(p-2a) > 0 \text{ and } p > |2a|.$$

Hence, on applying Theorem III, with $k = 0$, we obtain

$$\frac{\Gamma(\frac{1}{2}\nu \pm \frac{1}{2}m + \frac{5}{8})}{\sqrt{(2\pi)} \cdot \Gamma(\nu+1)} \cdot {}_2F_1\left\{\begin{matrix} \frac{1}{2}\nu \pm \frac{1}{2}m + \frac{5}{8} \\ \nu+1 \end{matrix}; \frac{4a^2}{p^2}\right\} \\ = \frac{n! \Gamma(m + \frac{1}{2})}{2^{2\nu+1}} \sum_{r=0}^n \frac{(-)^{r+n}}{(n-r)! r! \Gamma(m-n+r+\frac{1}{2})} \sum_{s=0}^{\infty} \frac{\Gamma(\nu+m+r+2s+\frac{5}{2})}{s! \Gamma(\nu+s+1)} \\ \times \frac{\Gamma(\nu-m+2s+\frac{5}{4})}{\Gamma(\nu+n+2s+\frac{7}{4})} \left(\frac{a}{2p}\right)^{2s} \cdot {}_2F_1\left\{\begin{matrix} \nu+m+r+2s+\frac{5}{4}, \nu-m+2s+\frac{5}{4} \\ \nu+n+2s+\frac{7}{4} \end{matrix}; \frac{1}{2}\right\} \quad (1)$$

$$\mathbf{R}(\frac{1}{2} + m) > 0, \mathbf{R}(\nu+m+\frac{5}{4}) > 0, \mathbf{R}(p-2a) > 0 \text{ and } p | > |2a|.$$

From (1), on comparing coefficients of p^{-2s} from both sides and replacing $\frac{1}{2}\nu + \frac{1}{2}m + \frac{5}{8}$, $\frac{1}{2}\nu - \frac{1}{2}m + \frac{5}{8}$ by α , β respectively, we obtain

$$\sum_{r=0}^n \frac{(-)^{r+n} \Gamma(2\alpha+r+2s)}{(n-r)! r! \Gamma(\alpha-\beta-n+r+\frac{1}{2})} \cdot {}_2F_1\left\{\begin{matrix} 2\alpha+r+2s, 2\beta+2s \\ \alpha+\beta+n+2s+\frac{1}{2} \end{matrix}; \frac{1}{2}\right\} \\ = \frac{2^{2\alpha+2s-1} \Gamma(\alpha+s) \Gamma(\alpha+\beta+n+2s+\frac{1}{2})}{n! \Gamma(\alpha-\beta+\frac{1}{2}) \Gamma(\beta+s+\frac{1}{2})}, \quad \mathbf{R}(\alpha+\frac{1}{2}) > \mathbf{R}(\beta) > 0 \text{ and } \mathbf{R}(\alpha) > 0.$$

8. **Theorem IV.** If

$$\varphi(p) \frac{\frac{1}{2} + m + \mu}{\pm m} f(x), \quad \mu \text{ being a positive integer}$$

and

$$\varphi_r(p) \frac{\frac{1}{2} + m + \frac{1}{2}r + \mu - n}{\pm(m + \frac{1}{2}r)} x^{\frac{1}{2}r} f(x),$$

then

$$\varphi(p) = n! \Gamma(2m + \mu + 1) \sum_{r=0}^n \frac{(-)^{r+n} (2p)^{\frac{1}{2}r}}{(n-r)! r! \Gamma(2m + \mu - n + r + 1)} \varphi_r(p)$$

* This is obtained by applying (R. 1). p. 19, in Example 11(a). p. 16.

provided $\mathbf{R}(\mu_1 + m + \frac{5}{4}) > 0$ where $f(x) = O(x^{\mu_1})$ for small x and $x^{m+5/4}e^{-px}L_{\mu-n}^{2m}(2p_0x)f(x)$ is bounded for $\mathbf{R}(p) > p_0 > 0$ and $x \geq 0$.

Proof: We have

$$(-)^r(2p)^{\frac{1}{2}r}\varphi_r(p) = (-)^{\mu-n+r}(\mu-n)!p \int_0^\infty (2xp)^{m+r+\frac{1}{2}}e^{-px}L_{\mu-n}^{2m+r}(2px)f(x)dx \quad (1)$$

On multiplying by $\frac{1}{(n-r)!r!\Gamma(2m+\mu-n+r+1)}$ and taking the sum from $r=0$ to $r=n$ and using (Hari Shanker 1942, p. 58),

$$\mu!L_{\mu}^{2m}(z) = n!(\mu-n)!\Gamma(2m+\mu+1) \sum_{r=0}^n \frac{(-z)^r L_{\mu-n}^{2m+r}(z)}{(n-r)!r!\Gamma(2m+\mu-n+r+1)},$$

we obtain

$$\sum_{r=0}^n \frac{(-)^r(2p)^{\frac{1}{2}r}}{(n-r)!r!\Gamma(2m+\mu-n+r+1)}\varphi_r(p) = \frac{(-)^n}{n!\Gamma(2m+\mu+1)}\varphi(p)$$

provided $\mathbf{R}(\mu_1 + m + \frac{5}{4}) > 0$ where $f(x) = O(x^{\mu_1})$ for small x and $x^{m+5/4}e^{-px}L_{\mu}^{2m}(2p_0x)f(x)$ is bounded for $\mathbf{R}(p) > p_0 > 0$ and $x \geq 0$.

9. *Example.* If

$$f(x) = x^\lambda,$$

then (Bose 1949, p. 12),

$$\varphi(p) = \frac{\Gamma_2(\lambda \pm m + \frac{5}{4})p^{-\lambda}}{2^{\lambda+1}\Gamma(\lambda-m-\mu+\frac{5}{4})} {}_2F_1\left\{\lambda \pm m + \frac{5}{4}; \frac{1}{2}\right\}, \mathbf{R}(p) > 0 \text{ and } \mathbf{R}(\lambda \pm m + \frac{5}{4}) > 0,$$

and

$$\varphi_r(p) = \frac{\Gamma(\lambda+m+r+\frac{5}{4})\Gamma(\lambda-m+\frac{5}{4})p^{-\lambda-\frac{1}{2}r}}{2^{\lambda+\frac{1}{2}r+1}\Gamma(\lambda-m-\mu+n+\frac{5}{4})} \times {}_2F_1\left\{\lambda+m+r+\frac{5}{4}, \lambda-m+\frac{5}{4}; \frac{1}{2}\right\}.$$

Hence, on applying Theorem IV, we obtain

$$\begin{aligned} & \frac{\Gamma(\lambda+m+\frac{5}{4})}{\Gamma(\lambda-m-\mu+\frac{5}{4})} {}_2F_1\left\{\lambda \pm m + \frac{5}{4}; \frac{1}{2}\right\} \\ &= \frac{n!\Gamma(2m+\mu+1)}{\Gamma(\lambda-m-\mu+n+\frac{5}{4})} \sum_{r=0}^n \frac{(-)^{r+n}\Gamma(\lambda+m+r+\frac{5}{4})}{(n-r)!r!\Gamma(2m+\mu-n+r+1)} \\ & \quad \times {}_2F_1\left\{\lambda+m+r+\frac{5}{4}, \lambda-m+\frac{5}{4}; \frac{1}{2}\right\}, \mathbf{R}(p) > 0 \text{ and } \mathbf{R}(\lambda \pm m + \frac{5}{4}) > 0. \end{aligned}$$

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UNION CURVES AND HYPER-ASYMPTOTIC CURVES ON THE SURFACE OF REFERENCE OF A RECTILINEAR CONGRUENCE

BY

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Union curves and dual union curves have been defined and studied in projective space by Sperry (1928). A curve on a surface is called a union curve of a congruence Γ in case the curve is such that its osculating plane at each of its points P contains the line l of Γ through P . Union curves were studied in an Euclidean space of three dimensions by Springer (1945) and Mishra (1950A). The curves which have the property that their rectifying planes at all their points contain the lines of the congruence through those points were studied by Mishra (1950B). Let us call these curves *hyper-asymptotic curves*. The reason why these curves have been so named is that they reduce to asymptotic lines when the congruence is formed by normals to the surface of reference. The object of this paper is to study union curves and hyper-asymptotic curves in further detail.

The notation generally adopted is that used by Eisenhart (1940) unless it is expressly stated.

1. Let the coordinates of a point P on the surface of reference S of a rectilinear congruence be given by $x^i = x^i(u^1, u^2)$, ($i = 1, 2, 3$) and the direction cosines of the ray l of the congruence through x^i by $\lambda^i = \lambda^i(u^1, u^2)$. Consider on the surface S any curve $C: x^i = x^i(s)$ at P . The direction of its tangent PT is determined by a value of du^2/du^1 . Let w be the angle which the vector λ^i makes with the principal normal to the curve at P . Then

$$\cos w = \rho \lambda^i \cdot \frac{d^2 x^i}{ds^2} \quad (1.1)$$

where ρ is the radius of curvature of the curve at P . But

$$\frac{d^2 x^i}{ds^2} = \rho^a x^i_{,a} + k_n X^i \quad (1.2)$$

where ρ^a are the components of the curvature vector of the curve C at P and k_n is the curvature of the normal section of the surface in the direction of the curve.

Use of (1.2) in (1.1) yields

$$\frac{\cos w}{\rho} = \lambda^i (\rho^a \tilde{x}^i_{,a} + k_n X^i) = \rho^a p_a + k_n q \quad (1.3)$$

where $p_a = \lambda^i x^i_{,a}$; and $\lambda^i X^i = \cos \theta = q$, θ being the angle which the normal to the surface at x^i makes with the line of the congruence through x^i .

The quantity $(\cos w)/\rho$ is the same for all curves through P having a common tangent at P denoted by PT . Consider in particular the plane curve in which the surface is cut by the plane determined by the line l at P and the tangent PT . Let us call it the *congruence section* of the surface for the direction PT . In this case w is 0° or 180° according as the principal normal to C and the line l have the same or opposite directions. If we denote by ρ_c the radius of curvature of this plane curve at P assumed to be positive, we have

$$\cos w/\rho = e/\rho_c$$

where e is $+1$ or -1 according as the principal normal to C and the line l have the same or opposite directions.

If we define a quantity k_e for a direction du^a at a point u^a by

$$k_e = \rho^a p_a + q k_n \quad (1.4)$$

it follows that the absolute value of k_e is the curvature of the congruence section of the surface for the given direction.

We know that the equation of the hyper-asymptotic curves is given by (Mishra, 1950B)

$$\rho^a p_a + q k_n = 0$$

hence a hyper-asymptotic curve may be characterised by the property that the curvature of the congruence section of the surface in the direction of the curve vanishes. This is reminiscent of the metric theorem that the curvature of the normal section of a surface in the direction of an asymptotic line is zero.

The curvature of a union curve is given by (Mishra 1950A)

$$k = \operatorname{cosec} \varphi (\rho^a p_a + k_n q)$$

which by virtue of (1.4) assumes the form

$$k = k_e \operatorname{cosec} \varphi.$$

From this equation it can be easily seen that the straight lines on a surface are the only union hyper-asymptotic lines. This is a generalisation of the result: the straight lines on a surface are the only geodesic asymptotic lines.

From the definition of hyper-asymptotic lines it is seen that the envelope of rectifying planes of these curves is the line l itself.

2. A union curve is a class of curves which have their differential equations of the type (Fubini 1918; Wilczynski 1922)

$$\frac{d^2 u^2}{du^1 du^1} = A + B \frac{du^2}{du^1} + C \left(\frac{du^2}{du^1} \right)^2 + D \left(\frac{du^2}{du^1} \right)^3$$

where the coefficients are functions of u^1 and u^2 . These curves are called hypergeodesics. We now find the values of A, B, C, D in the case of union curves.

The equation of a union curve is given by (Springer 1945)

$$e_{\alpha\beta}(k_n p^\alpha - q p^\alpha) du^\beta = 0 \quad (2.1)$$

where

$$e_{\alpha\beta} = (X^i x^i_{,\alpha} x^i_{,\beta}) \text{ and } p^\alpha = g^{\alpha\beta} p_\beta.$$

Expanding (2.1) and observing that

$$\frac{du^2}{ds} = \frac{du^2}{du^1} \frac{du^1}{ds}, \quad \frac{d^2u^2}{ds^2} = \frac{d^2u^2}{du^1 du^1} \left(\frac{du^1}{ds}\right)^2 + \frac{du^2}{du^1} \frac{d^2u^1}{ds^2}$$

we get,

$$\begin{aligned} \frac{d^2u^2}{du^1 du^1} = & h^2 d_{11} - \left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} + \left(\left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} - 2 \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} + 2d_{12}h^2 - d_{11}h^1 \right) \frac{du^2}{du^1} \\ & + \left(2 \left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} - \left\{ \begin{matrix} 2 \\ 22 \end{matrix} \right\} + d_{22}h^2 - 2d_{12}h^1 \right) \left(\frac{du^2}{du^1} \right)^2 + \left(\left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} - d_{22}h^1 \right) \left(\frac{du^2}{du^1} \right)^3 \end{aligned} \quad (2.2)$$

where $h^a = p^a/q$.

The values of A, B, C, D are indicated by this equation. In particular if the congruence is formed by normals to the surface of reference $h^a = 0$, and we get the equation of a geodesic in the form (Eisenhart 1909, p. 205),

$$\frac{d^2u^2}{du^1 du^1} = - \left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} + \left(\left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} - 2 \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} \right) \frac{du^2}{du^1} + \left(2 \left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} - \left\{ \begin{matrix} 2 \\ 22 \end{matrix} \right\} \right) \left(\frac{du^2}{du^1} \right)^2 + \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} \left(\frac{du^2}{du^1} \right)^3$$

The equation of a dual union curve on S (Lane 1932, p. 103) is given by

$$\begin{aligned} \frac{d^2u^2}{du^1 du^1} = & \left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} - h^2 d_{11} + \left(\left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} - 2 \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} + 2d_{12}h^2 - d_{11}h^1 \right) \frac{du^2}{du^1} \\ & + \left(2 \left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} - \left\{ \begin{matrix} 2 \\ 22 \end{matrix} \right\} + d_{22}h^2 - 2d_{12}h^1 \right) \left(\frac{du^2}{du^1} \right)^2 + \left(d_{22}h^1 - \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} \right) \left(\frac{du^2}{du^1} \right)^3 \end{aligned} \quad (2.3)$$

The directions of Segrè are characterised as the directions in which the union curves and dual union curves coincide (Lane 1932, p. 116). From (2.2) and (2.3) it is easily seen that if on a surface union curves and dual union curves coincide, then the differential equation of the curves of Segrè assumes the form

$$\left(\left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} - h^2 d_{11} \right) (du^1)^3 + \left(d_{22}h^1 - \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} \right) (du^2)^3 = 0$$

and the curves of Darboux are given by

$$\left(\left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} - h^2 d_{11} \right) (du^1)^3 + \left(\left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} - h^1 d_{22} \right) (du^2)^3 = 0$$

From (2.2) it is seen that if a parametric curve $u^2 = \text{constant}$ is a union curve, then

$$h^2 d_{11} - \left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} = 0$$

which by (2.3) is the condition that the curve is a dual union curve also. But in that case the curve will be a line of Segrè. Hence if a parametric curve is a union curve or a dual union curve it must be a line of Segrè.

We now find the condition for a hyper-asymptotic curve to be a hypergeodesic.

The equation of a hyper-asymptotic curve is given by

$$\rho^a p_a + q k_\eta = 0$$

Expanding this equation we get ;

$$\frac{d^2 u^1}{ds^2} \left(p_1 + \frac{du^2}{du^1} p_2 \right) + \left(\frac{du^1}{ds} \right)^2 \left[\left(\left\{ \begin{smallmatrix} 1 \\ 11 \end{smallmatrix} \right\} + 2 \left\{ \begin{smallmatrix} 1 \\ 12 \end{smallmatrix} \right\} \frac{du^2}{du^1} + \left\{ \begin{smallmatrix} 1 \\ 22 \end{smallmatrix} \right\} \left(\frac{du^2}{du^1} \right)^2 \right) p_1 \right. \\ \left. + \left(\frac{d^2 u^2}{du^1 du^1} + \left\{ \begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right\} + 2 \left\{ \begin{smallmatrix} 2 \\ 12 \end{smallmatrix} \right\} \frac{du^2}{du^1} + \left\{ \begin{smallmatrix} 2 \\ 22 \end{smallmatrix} \right\} \left(\frac{du^2}{du^1} \right)^2 \right) p_2 + \left(d_{11} + 2d_{12} \frac{du^2}{du^1} + d_{22} \left(\frac{du^2}{du^1} \right)^2 \right) p_1 \right] = 0$$

This represents a hypergeodesic if $p_a du^a = 0$ which is true when either the congruence is formed by normals to the surface of reference in which case the hyper-asymptotic lines reduce to asymptotic lines or the lines of the congruence are tangent to one parameter family of curves.

3. Referred to asymptotic lines the direction cosines of a projective normal through x^i are given by (Lane 1932, p. 248)

$$\frac{(\log \lambda k^{\dagger})_{,2} x^1_{,1} + (\log \lambda k^{\dagger})_{,1} x^1_{,2} + d_{12} X^i}{\{[(\log \lambda k^{\dagger})_{,2}]^2 g_{11} + [(\log \lambda k^{\dagger})_{,1}]^2 g_{22} + 2(\log \lambda k^{\dagger})_{,1} (\log \lambda k^{\dagger})_{,2} g_{12} + d_{12}^2\}^{\frac{1}{2}}}$$

where

$$\lambda = \frac{\left(\left\{ \begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} 1 \\ 22 \end{smallmatrix} \right\} \right)^{\frac{1}{2}}}{(d_{12} e_{12})^{\frac{1}{2}}} \text{ and } k = \frac{e_{12}}{d_{12}}.$$

Let the denominator of the expression for these direction cosines be denoted by A , then for a congruence formed by projective normals

$$p_a = \frac{(\log \lambda k^{\dagger})_{,2} g_{1a} + (\log \lambda k^{\dagger})_{,1} g_{2a}}{A} \text{ and } q = \frac{d_{12}}{A}.$$

Hence the equation of the hyper-asymptotic lines is given by

$$\rho^a [(\log \lambda k^{\dagger})_{,2} g_{1a} + (\log \lambda k^{\dagger})_{,1} g_{2a}] + d_{12} k_n = 0$$

and the equation of union curves is given by

$$e_{\alpha\beta} k_n g^{\alpha\gamma} [(\log \lambda k^{\dagger})_{,1} g_{1\gamma} + (\log \lambda k^{\dagger})_{,2} g_{2\gamma}] du^{\beta} - d_{12} k_g = 0$$

where k_g is the geodesic curvature of the curve.

From these equations a number of properties of union curves and hyper-asymptotic curves of a projective normal congruence can be found.

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CONVERGENCE OF RANDOM DISTRIBUTION FUNCTIONS

By

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Convergence of sequences of probability distribution functions have been considered in statistical literature, the most important theorem being the second limiting theorem of Frechet & Shohat (1931), which demonstrates that under certain conditions the convergence of moments implies the convergence of the distribution functions. In many statistical problems, *e.g.*, in the theory of nonparametric tests we have to consider sequences of distribution functions depending upon random elements. We shall in this paper consider the convergence of a sequence of distribution functions depending upon a sequence of random variables $\xi_1, \xi_2, \dots, \xi_n, \dots$ dependent or not.

Let $F_n(x, \xi_1, \dots, \xi_n)$ be a distribution function depending on the random variables ξ_1, \dots, ξ_n and $\phi_n(t, \xi_1, \dots, \xi_n)$ the corresponding characteristic function. Let

$$\mu_r^{(n)} = \int_{-\infty}^{\infty} x^r dF_n(x, \xi_1, \dots, \xi_n)$$

be the moments of $F_n(x, \xi_1, \dots, \xi_n)$. $\mu_r^{(n)}$ thus a random variable depending upon ξ_1, \dots, ξ_n .

Let

$$\text{Plim}_{n \rightarrow \infty} \mu_r^{(n)} = \mu_r$$

i.e., $\mu_r^{(n)}$ converges to μ_r in probability, the notation being due to Wald & Mann (1943).

We have

$$\phi_n(t) = \phi_n(t, \xi_1, \dots, \xi_n) = \sum_{r=1}^k \mu_r^{(n)} \frac{(it)^r}{r!} + \frac{(it)^{k+1}}{(k+1)!} \frac{d^{k+1}}{dz^{k+1}} [\phi_n(z)]_{z=it}$$

$$\phi(t) = \sum \mu_r \frac{(it)^r}{r!} + \frac{(it)^{k+1}}{(k+1)!} \frac{d^{k+1}}{dz^{k+1}} [\phi(z)]_{z=\theta'it}, \quad 0 < |\theta| < 1, \quad 0 < |\theta'| < 1.$$

Thus

$$|\phi_n(t) - \phi(t)| \leq \sum_{r=1}^k \frac{|t|^r}{r!} \text{Max}_{r \leq k} |\mu_r^{(n)} - \mu_r| + \frac{|t|^{k+1}}{(k+1)!} \left\{ \text{Max} \left| \frac{d^{k+1}}{dz^{k+1}} \phi_n(z) \right| + \text{Max} \left| \frac{d^{k+1}}{dz^{k+1}} \phi(z) \right| \right\}$$

Now

$$\frac{d^{k+1}}{dz^{k+1}} \phi_n(z) = i^{k+1} \int_{-\infty}^{\infty} x^{k+1} e^{izx} dF_n(x)$$

thus

$$\left| \frac{d^{k+1}}{dz^{k+1}} \phi_n(z) \right| \leq v_{k+1}^{(n)}$$

where $\nu_{k+1}^{(n)}$ is the absolute moment of order $k+1$ for $F_n(x)$. Thus

$$|\phi_n(t) - \phi(t)| \leq \sum_{r=1}^k \frac{|t|^r}{r!} \text{Max}_{r \leq k} |\mu_r^{(n)} - \mu_r| + \frac{|t|^{k+1}}{(k+1)!} \{\nu_{r+1} + \nu_{k+1}^{(n)}\}.$$

From Plim $\mu_r^{(n)} = \mu_r$,

$$\text{Prob}\{|\mu_r^{(n)} - \mu_r| > \varepsilon\} < \delta_1 \text{ for } n > n_r \text{ holds.}$$

Thus

$$\text{Prob}\{|\mu_r^{(n)} - \mu_r| < \varepsilon; r = 1, \dots, k+1\} > 1 - (k+1)\delta_1, \\ \text{for } n > \text{Max}(n_1, \dots, n_{k+1}) = N \text{ say.}$$

When $k+1$ is even $\mu_{k+1} = \nu_{k+1}$, $\mu_{k+1}^{(n)} = \nu_{k+1}^{(n)}$ and $(\nu_{k+1} + \nu_{k+1}^{(n)}) < 2\nu_{k+1} + \varepsilon$.

Thus when $n > N$, except for a set of values of (ξ_1, \dots, ξ_n) of probability less than $(k+1)\delta_1$

$$|\phi_n(t) - \phi(t)| \leq \varepsilon \sum_{r=1}^k \frac{|t|^r}{r!} + \frac{|t|^{k+1}}{(k+1)!} \{2\nu_{k+1} + \varepsilon\} \leq \varepsilon[e^{|t|} - 1] + \frac{|t|^{k+1}}{(k+1)!} \{2\nu_{k+1} + \varepsilon\}$$

Consider the functions

$$G_n(x) = \frac{1}{h} \int_x^{x+h} F_n(z) dz, \quad G(x) = \frac{1}{h} \int_x^{x+h} F(z) dz.$$

As shown by Cramer (1933) the functions $G_n(x)$ and $G(x)$ are distribution functions and the characteristic function of $G_n(x)$ is given by

$$\phi_n(t) = \frac{1 - e^{-ith}}{ith}.$$

From the inversion theorem

$$G_n(x) - G_n(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_n(t) \frac{1 - e^{-ith}}{ith} \cdot \frac{1 - e^{-itx}}{it} dt$$

and

$$G(x) - G(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t) \frac{1 - e^{-ith}}{ith} \cdot \frac{1 - e^{-itx}}{it} dt$$

$$[G_n(x) - G(x)] - [G_n(0) - G(0)] = \frac{1}{2\pi h} \int_{-\infty}^{\infty} [\phi_n(t) - \phi(t)] \frac{1 - e^{-ith}}{it} \cdot \frac{1 - e^{-itx}}{it} dt$$

$$|[G_n(x) - G(x)] - [G_n(0) - G(0)]| \leq \frac{1}{2\pi h} \left| \int_{|t| \leq a} [\phi_n(t) - \phi(t)] \frac{(1 - e^{-ith})(1 - e^{-itx})}{-t^2} dt \right| \\ + \frac{1}{2\pi h} \int_{|t| > a} \frac{4|\phi_n(t) - \phi(t)|}{t^2} dt$$

since

$$\frac{1 - e^{-iz}}{iz} = 1 - \frac{iz}{2!} + \frac{(iz)^2}{3!} - \dots$$

$$\left| \frac{1 - e^{-iz}}{iz} \right| \leq 1 + |z| + \frac{|z|^2}{2!} + \dots \leq e^{|z|} \leq e, \text{ when } |z| < 1.$$

Also

$$\left| \frac{1 - e^{-iz}}{iz} \right| \leq 2, \text{ when } |z| \geq 1.$$

Thus $\left| \frac{1 - e^{-iz}}{iz} \right| \leq e$ holds for all values of z .

$$\begin{aligned} \left| \int_{|t| < \epsilon} [\phi_n(t) - \phi(t)] \frac{(1 - e^{-it\alpha})(1 - e^{-itx})}{ht^2} dt \right| &\leq |x| e^2 \int_{|t| < \epsilon} |\phi_n(t) - \phi(t)| dt \\ &\leq 2|x| e^2 [e e^\alpha + \frac{\alpha^{k+2}}{(k+2)!} \{2\nu_{k+1} + \epsilon\}] \end{aligned}$$

and

$$\int_{|t| > \epsilon} \frac{|\phi_n(t) - \phi(t)|}{t^2} dt \leq 2 \int_{|t| > \epsilon} \frac{dt}{t^2} \leq \frac{4}{\alpha}.$$

Thus we have

$$\begin{aligned} &|[G_n(x) - G(x)] - [G_n(0) - G(0)]| \\ &\leq \frac{8}{\pi h \alpha} + \frac{M e^2}{\pi} \{e e^\alpha + \frac{\alpha^{k+2}}{(k+2)!} [2\nu_{k+1} + \epsilon]\}, \text{ for } |x| < M. \end{aligned}$$

We shall make the following assumption.

Assumption A. $\lim_{k \rightarrow \infty} \frac{\alpha^{k+2} \nu_{k+1}}{(k+2)!} = 0$ for any given value of α .

It may be recalled that this is a sufficient condition that the set of moments should determine a distribution uniquely. Now, when assumption A holds for any given value of h and for $|x| < M$.

We may select α so large that $8/\pi h \alpha < \frac{1}{2} \epsilon_1$ for any given value of α we may choose k so large that $\frac{M e^2}{\pi} \frac{\alpha^{k+2}}{(k+2)!} [2\nu_{k+1} + \epsilon] < \frac{\epsilon_1}{5}$ and finally n so large that $\frac{M e^2}{\pi} e e^\alpha < \frac{\epsilon_1}{8}$ and $(k+1)\delta_1 < \delta$. Thus for $n > n_0(h, M)$

$$|[G_n(x) - G(x)] - [G_n(0) - G(0)]| < \epsilon_1$$

$$(2) \text{ i.e., } |[G_n(x) - G_n(x)] - [G_n(-h) - G(-h)]| < 2\epsilon_1$$

holds with probability greater than $1 - \delta$ where ϵ_1 and δ are arbitrarily small.

Again, since $F_n(z)$ is an increasing function of z

$$\begin{aligned} \frac{1}{h} \int_x^{x+h} [F_n(z) - F(z)] dz &\geq F_n(x) - \frac{1}{h} \int_x^{x+h} F(z) dz \\ &\geq [F_n(x) - F(x)] - \frac{1}{h} \int_x^{x+h} [F(z) - F(x)] dz \\ \frac{1}{h} \int_{-h}^0 [F_n(z) - F(z)] dz &\leq F_n(0) - \frac{1}{h} \int_{-h}^0 F(z) dz \\ &\leq [F_n(0) - F(0)] - \frac{1}{h} \int_{-h}^0 [F(z) - F(0)] dz. \end{aligned}$$

Subtracting the second inequality from the first we have

$$\begin{aligned}
 [G_n(x) - G(x)] - [G_n(-h) - G(-h)] &\geq [F_n(x) - F(x)] - [F_n(0) - F(0)] \\
 &\quad - \frac{1}{h} \int_x^{x+h} [F(z) - F(x)] dz + \frac{1}{h} \int_{-h}^0 [F(z) - F(0)] dz \\
 \text{i.e., } [F_n(x) - F(x)] - [F_n(0) - F(0)] &\leq |[G_n(x) - G(x)] - [G_n(-h) - G(-h)]| \\
 &\quad + \frac{1}{h} \left| \int_{-h}^0 [F(z) - F(0)] dz \right| + \frac{1}{h} \left| \int_x^{x+h} [F(z) - F(x)] dz \right|
 \end{aligned}$$

Now when $dF(z)/dz$ exists and is bounded

$$(3) \quad \frac{1}{h} \int_x^{x+h} [F(z) - F(x)] dz = O(h).$$

Thus for h sufficiently small

$$[F_n(x) - F(x)] - [F_n(0) - F(0)] \leq 2\epsilon_1 + \epsilon_2$$

Let $\bar{F}_n(x) = 1 - F_n(-x)$ and $\bar{F}(x) = 1 - F(-x)$, then $\bar{F}_n(x)$ and $\bar{F}(x)$ are the distribution functions with moments $\mu_r^{(n)}$ and μ_r respectively and being the distribution functions of $(-x)$ in either case. Since all the above arguments hold for these distribution functions we have

$$\begin{aligned}
 [\bar{F}_n(x) - \bar{F}(x)] - [\bar{F}_n(0) - \bar{F}(0)] &\leq |[\bar{G}_n(x) - \bar{G}(x)] - [\bar{G}_n(-h) - \bar{G}(-h)]| \\
 &\quad + \frac{1}{h} \left| \int_h^{x+h} [\bar{F}(z) - \bar{F}(x)] dz \right| + \frac{1}{h} \left| \int_{-h}^0 [\bar{F}(z) - \bar{F}(0)] dz \right|
 \end{aligned}$$

where $\bar{G}_n(x)$ and $\bar{G}(x)$ correspond to $\bar{F}_n(x)$ and $\bar{F}(x)$ respectively.

$$\begin{aligned}
 \text{i.e., } -[F_n(-x) - F(-x)] + [F_n(0) - F(0)] &\leq 2\epsilon_1 + \epsilon_2 \\
 -[F_n(x) - F(x)] + [F_n(0) - F(0)] &\leq 2\epsilon_1 + \epsilon_2.
 \end{aligned}$$

Thus we have

$$(4) \quad |[F_n(x) - F(x)] - [F_n(0) - F(0)]| \leq 2\epsilon_1 + \epsilon_2$$

for $h < h_1$ and $n > N_0(h_1, M)$ and $|x| < M$.

Again, from Tchebyscheff's inequality

$$(5) \quad 1 - F_n(x) = \int_x^\infty dF_n(z) \leq \frac{\mu_2^{(n)}}{x^2} \leq \frac{\mu_2 + \epsilon}{x^2} < \epsilon_3$$

$$(6) \quad 1 - F(x) = \int_x^\infty dF(z) \leq \frac{\mu_2}{x^2} < \epsilon_4,$$

since $\text{Prob}\{|\mu_2^{(n)} - \mu_2| < \epsilon\} > 1 - \delta$, when $n > N_1$ and when x is sufficiently large.

Thus except for values of (ξ_1, \dots, ξ_n) of probability less than δ_2

$$|F_n(x) - F(x)| < \epsilon_3 + \epsilon_4 \text{ for } |x| \geq M.$$

Thus we have

$$|F_n(0) - F(0)| < 2\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = \epsilon' \text{ say}$$

for $n \geq N_0(h_1, M)$ and $n > N_1$.

Hence we have

$$|F_n(x) - F(x)| < 2\epsilon_1 + \epsilon_2 + \epsilon' < \epsilon''$$

for $n > N_0(h, M)$, N_1 except for values of (ξ_1, \dots, ξ_n) of probability less than, $\delta_1 + \delta$, ϵ , δ , δ_2 being arbitrarily small.

Thus we may state:

Theorem A. Let $F_n(x; \xi_1, \dots, \xi_n)$ be a sequence of distribution functions depending on a set of random variables $\xi_1, \dots, \xi_n, \dots$ such that

$$\text{Plim}_{n \rightarrow \infty} \mu_r^{(n)} = \mu_r, \quad (r = 1, 2, \dots)$$

where $\mu_r^{(r)}$ is the r -th moment of $F_n(x)$ and μ_r is the r -th moment of a distribution $F(x)$ which possesses a density function and for which assumption (A) holds, then for ϵ and δ arbitrarily small, $|F_n(x) - F(x)| < \epsilon$ for $n > N(\epsilon, \delta)$ except for values ξ_1, \dots, ξ_n with probability less than δ .

The convergence of $F_n(x)$ considered in theorem A is a case of convergence in probability i.e., convergence in the weak sense we shall prove the following theorem for the strong convergence of the random distribution function.

Theorem B. A sufficient condition for the almost certain convergence of the random distribution function $F_n(x, \xi_1, \dots, \xi_n)$ to $F(x)$ is that

$$\text{Prob} \left\{ \lim_{n \rightarrow \infty} \mu_r^{(n)} = \mu_r \right\} = 1$$

hold for $r = 1, 2, 3, \dots$

Proof. Since ξ_i are real random variables, dependent or not, the joint distribution function of ξ_1, \dots, ξ_n defines a probability distribution over a class of Borel sets in the infinite dimensional space of $\xi_1, \xi_2, \dots, \xi_n, \dots$. For the purpose we consider the cylinder sets of Kolmogoroff as the Borel sets. These cylinder sets consists of all sequences

$$\xi_1, \dots, \xi_n, \dots$$

for which the first k variables lie in specified sets in the space of (ξ_1, \dots, ξ_n) , the probabilities of the cylinder sets being the probabilities of (ξ_1, \dots, ξ_n) lying in given sets. As shown by Kolmogoroff (1933) this procedure defines a probability distribution in the infinite dimensional space as a totally additive set function.

From the equation

$$P \left\{ \lim_{n \rightarrow \infty} \mu_r^{(n)} = \mu_r \right\} = 1$$

we see that except for a set A_r of points $(\xi_1, \dots, \xi_k, \dots)$ in the infinite dimensional space of probability ϵ^r

$$|\mu_r^{(n)} - \mu_r| < \eta_n^{(r)} \quad \text{for } n > n_r$$

where $\eta_n^{(r)}$ are nonincreasing sequences with limit zero. This follows from the property that a random variable converging almost certainly also converges uniformly in probability (Frechet, 1937).

Thus the moments $\mu_r^{(n)}$ converge to μ_r except for sequences $(\xi_1, \dots, \xi_k, \dots)$ belonging to

$$A = A_1 \cup A_2 \cup \dots \cup A_n \cup \dots$$

where $|A| \geq \sum_{n=1}^{\infty} \epsilon^n = 2\epsilon$. From the second limiting theorem of Frechet and Shohat (1931), $F_n(x)$ converges to $F(x)$ except for sequences in A . Since ϵ is arbitrarily small it follows that $F_n(x)$ converges almost certainly to $F(x)$. Since $F_n(x)$ is continuous $F_n(x)$ converges uniformly to $F(x)$ with respect to x . Hence the theorem

Applications. 1. Let x_1, \dots, x_n, \dots be a sequence of independent random variables with the same distribution function $F(x)$. Corresponding to the samples x_1, \dots, x_n consider the sample distribution function $S_n(x)$ (or repartition function according to Von Mises). If n_x is the number of values x_1, \dots, x_n less than x then

$$S_n(x) = n_x/n,$$

we have thus

$$\int_{-\infty}^{\infty} x^r dS_n(x) = \frac{x_1^r + \dots + x_n^r}{n}.$$

Thus when all the moments of $F(x)$ exists we have

$$P \left\{ \lim_{n \rightarrow \infty} \frac{x_1^r + \dots + x_n^r}{n} = E(x^r) \right\} = 1$$

from the strong law of large numbers. Hence the conditions of Theorem B are fulfilled as $S_n(x)$ converges almost certainly to $F(x)$.

When the moments of $F(x)$ do not exist we can find a distribution function $H(x)$ with finite bounds such that

$$|F(x) - H(x)| < \epsilon$$

where ϵ is arbitrarily small. In particular, let

$$H(x) = \frac{F(x)}{F(\beta) - F(\alpha)} \quad \text{when } \alpha \leq x \leq \beta$$

$$= 0 \quad \text{otherwise.}$$

Let $S_n(x, H)$ be the repartition of a sample from $H(x)$ obtained by removing values of $x < \alpha$ and $x > \beta$ from a sample from $F(x)$.

The moments of $H(x)$ are naturally finite and thus from Theorem B $S_n(x, H)$ converges almost certainly to $H(x)$. Now

$$S_n(x, H) = \frac{n_x - n_\alpha}{n - n_\alpha} = \frac{n}{n_\beta - n_\alpha} [S_n(x) - S_n(\alpha)]$$

Since $(n_\beta - n_\alpha)/n$ converges almost certainly to $F(\beta) - F(\alpha)$, it follows that $S_n(x) - S_n(\alpha)$ converges almost certainly to $F(x) - F(\alpha)$ for x and α in (α, β) . Since α, β are arbitrary it follows that $S_n(x)$ converges almost certainly to $F(x)$ uniformly for x . Thus we state

Theorem C. When x_1, \dots, x_n, \dots a sequence of independent random variables with the same distribution function $F(x)$ then

$$|S_n(x) - F(x)| < \epsilon''(n)$$

for $n = n_0, n_0 + 1, \dots$ with probability greater than $1 - \delta$, $\epsilon''(n)$ being a nonincreasing sequence of n such that $\lim_{n \rightarrow \infty} \epsilon''(n) = 0$.

This is the celebrated theorem of Glivenko-Cantelli (Glivenko, 1938; Cantelli, 1935), and has been called by Cantelli as the fundamental theorem of statistics. The method of proof given above may be useful for the case where the variables are dependent, although in the case of independence the proof is not simpler than the one given by Frechet (1937).

2. Wald and Wolfowitz (1943) have shown that the nonparametric distribution of the serial correlation coefficient

$$T = \frac{x_1 x_2 + \dots + x_n x_1}{\sqrt{n}}$$

converges asymptotically to the normal distribution when the sequence satisfies the relations

$$(C) \quad \frac{\mu_r}{\mu_2^{r/2}} = O(1), \quad r = 3, 4, \dots,$$

where

$$\mu_r = \frac{1}{n} \sum_{a=1}^n \left(x_a - \frac{1}{n} \sum_{a=1}^n x_a \right)^r$$

As noticed by Wald and Wolfowitz (1943), this condition is satisfied with probability one, when x_1, \dots, x_n, \dots are independent random variables with the same distribution function $F(x)$ with finite moments. We know from the strong law of large numbers that when the moments of all order exist the relations (C) hold in a strong sense. Since whenever the condition (C) holds the moments of the nonparametric distribution converge to the moments of a normal distribution converge to the moments of a normal distribution, it follows that the distribution of the statistic T converges almost certainly to the normal distribution.

However, the condition (C) may be relaxed as shown by G. E. Noether (1949) and we consider

$$(C_1) \quad \frac{\sum (x_i - \bar{x})^r}{\{\sum (x_i - \bar{x})\}^{r/2}} = O(n^{\frac{2-r}{4}})$$

when (C_1) holds the moments of the nonparametric distribution of T converge to the moments of a normal distribution. It is easy to show that the condition (C_1) holds when $\{x_i\}$ are independent random variables with the same distribution function $G(x)$ for which absolute moments of order $4 + \delta$ exist.

We shall assume $E(x_i) = 0$ without loss of generality. From Minkowsk's inequality

$$\{(x_1 - \bar{x})^r + \dots + (x_n - \bar{x})^r\} \leq [(x_1 - \bar{x})^4 + \dots + (x_n - \bar{x})^4]^{r/4} \text{ for } \frac{1}{4}r < 1$$

Again when absolute moments of order $4 + \delta$ exist

$$\text{Prob} \left\{ \left| \frac{\sum x_i^4}{n} - \mu_4 \right| > \epsilon \right\} < \frac{k(\epsilon)}{n^{\delta/4}}$$

from Markoff's law of large numbers (Uspensky, 1937) and

$$\text{Prob} \left\{ \left| \frac{\sum x_i}{n} \right| > c \right\} < \frac{\mu_2}{nc^2}$$

Thus

$$\text{Prob} \left\{ \left| \frac{\sum (x_i - \bar{x})^4}{n} - \mu_4 \right| > \epsilon' \right\} < \frac{k'(\epsilon')}{n^{3/4}}.$$

Thus

$$\text{Prob} \{ (x_1 - \bar{x})^r + \dots + (x_n - \bar{x})^r \} \leq n^{1-r} A.$$

with probability greater than $1 - \frac{k}{n^{3/4}}$. Also by the same argument

$$\sum (x_i - \bar{x})^2 \geq n [\mu_2 - \epsilon'']$$

with probability greater than $1 - \frac{k_2}{n}$. Thus

$$\frac{\sum (x_i - \bar{x})^r}{\{\sum (x_i - \bar{x})^2\}^{r/2}} \leq k n^{-1/r}$$

with probability greater than $1 - \frac{k'}{n^{3/4}}$.

Thus the condition (C_1) holds with probability approaching unity. In this case, however, unless moments of order 8 exist, the strong law need not hold for

$$\frac{x_1^4 + \dots + x_n^4}{n}$$

and thus the conditions (C_1) hold only in a weak sense. Therefore in such a case theorem A holds and $F_n(x)$ converges to $F(x)$ in a weak sense.

B. Convergence of a sequence of random functions. Let $f_1(x, \alpha), \dots, f_n(x, \alpha), \dots$ be a sequence of random functions defined by a set of stochastic processes, where α is the element of a space P , over which a probability measure is defined. Glivenko (1938) has considered the convergence of a sequence of random functions to a non-random function as n increases. Let

$$F_n(x, \alpha) = \int_a^x |f_n(t, \alpha)| dt / \int_a^b |f_n(t, \alpha)| dt$$

where the integrals exist whenever the correlation function of the stochastic process $|f_n(t, \alpha)|$ is continuous.

$F_n(x)$ is thus a random distribution function defined for all values of $\alpha \in P$ for which

$$\int_a^b |f_n(t, \alpha)| dt > 0.$$

Instead of a sequence of random variables (ξ_1, \dots, ξ_n) we may naturally consider a sequence $(\alpha_1, \dots, \alpha_n)$ of elements of P and the Theorem B can be applied to the random functions $F_n(x, \alpha)$ whenever

$$P \left\{ \int_a^b |f_n(t, \alpha)| dt = 0 \right\} = 0.$$

Thus

$$\mu_r^{(n)} = \int_a^b x^r dF_n(x)$$

exists and is a function of α and when

$$\text{Plim } \mu_r^{(n)} = \mu_r = \int_a^b x^r dF(x)$$

from Theorem A

$$|F_n(x) - F(x)| > \epsilon \text{ for } n > n_0$$

with probability greater than $1 - \delta$.

Let

$$g_n(t, \alpha) = |f_n(t, \alpha)| / \int_a^b |f_n(t, \alpha)| dt$$

and

$$\lim_{n \rightarrow \infty} E\{g_n(t, \alpha)\} = g(t), \quad \mu_r = \int_a^b x^r g(x) dx.$$

(M)

$$\lim_{n \rightarrow \infty} E\{[\mu_r^{(n)} - \mu_r]^2\} = 0$$

implies that the random variable $\mu_r^{(n)}$ converges in probability to μ_r i.e., $\text{Plim } \mu_r^{(n)} = \mu_r$.

Now

$$E\{[\mu_r^{(n)} - \mu_r]^2\} = E\{[\mu_r^{(n)} - E(\mu_r^{(n)})]^2\} + [E(\mu_r^{(n)}) - \mu_r]^2$$

and

$$\lim_{n \rightarrow \infty} E(\mu_r^{(n)}) = \lim_{n \rightarrow \infty} E\left\{\int_a^b t^r [g_n(t, \alpha)] dt\right\} = \lim_{n \rightarrow \infty} \int_a^b g(t) dt$$

since the operator E is commutative with integration with respect to t from a generalisation of Fubini's theorem to abstract product spaces. Thus

$$\lim_{n \rightarrow \infty} [E(\mu_r^{(n)}) - \mu_r]^2 = 0.$$

$$E\{[\mu_r^{(n)} - E(\mu_r^{(n)})]^2\} = E\left[\left\{\int_a^b t^r g_n(t, \alpha) dt - \int_a^b t^r \bar{g}_n(t) dt\right\}^2\right], \text{ [where } \bar{g}_n(t) = E\{g_n(t, \alpha)\}]$$

$$= E\left[\left\{\int_a^b t^r [g_n(t, \alpha) - \bar{g}_n(t)] dt\right\}^2\right] = E\left[\int_a^b t^r dt \int_a^b s^r ds [g_n(t, \alpha) - \bar{g}_n(t)] [g_n(s, \alpha) - \bar{g}_n(s)]\right]$$

$$= \int_a^b t^r dt \int_a^b s^r ds E\{[g_n(t, \alpha) - \bar{g}_n(t)] [g_n(s, \alpha) - \bar{g}_n(s)]\} = \int_a^b dt \int_a^b (st)^r \Gamma_n(s, t) ds dt$$

where $\Gamma_n(s, t)$ is the correlation function of the stochastic process $g_n(x, \alpha)$.

Thus when

$$\int_a^b \int_a^b (st)^r \Gamma_n(s, t) ds dt$$

converges to zero as $n \rightarrow \infty$ (M) holds and thus $F_n(x, \alpha)$ converges to $F(x)$ in probability.

Now, when $\Gamma_n(s, t)$ is continuous, the stochastic process $g_n(t, \alpha)$ is continuous in mean square and

$$\int_a^x g_n(t, \alpha) dt$$

is differentiable in mean square (Loeve, 1947), then

$$\text{Plim } g_n(x, \alpha) = g(x), \quad \text{Plim } \frac{|f_n(x, \alpha)|}{\chi_n(\alpha)} = g(x)$$

where

$$\chi_n(\alpha) = \int_a^b |f_n(t, \alpha)| dt$$

Hence

$$\text{Plim } \left| \frac{f_n(x, \alpha)}{f_n(k, \alpha)} \right| = \frac{g(x)}{g(k)}$$

when the stochastic process starts from the point a , so that the random function $f_n(x, \alpha)$ has the value λ at the point $x = a$

$$\text{Plim } |f_n(x, \alpha)| = \lambda g(x).$$

We thus have the following theorem.

Theorem D. Let $\{|f_n(x, \alpha)|\}$ be a sequence of random functions and let $\Gamma_n(s, t)$ be the correlation of the stochastic process

$$|f_n(x, \alpha)| \bigg/ \int_a^b |f_n(t, \alpha)| dt$$

be continuous and

$$\lim_{n \rightarrow \infty} \int_a^b \int_a^b (st)^r \Gamma_n(s, t) ds dt = 0$$

for all integral values of r and let

$$\lim_{n \rightarrow \infty} E \left\{ |f_n(x, \alpha)| \bigg/ \int_a^b |f_n(t, \alpha)| dt \right\} = g(x)$$

then $\left| \frac{f_n(x, \alpha)}{f_n(k, \alpha)} \right|$ converges in probability to the non-random function $g(x)/g(k)$ in the interval (a, b) .

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EQUILIBRIUM OF ROTATING FLUID-BODIES IN CONFOCAL STRATIFICATIONS—I

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INTRODUCTION

The problem of rotating equilibrium of fluid-bodies in confocal stratifications was studied by Dive (1930). He proved the existence of such forms of equilibrium under self-gravitation and also obtained the remarkable result that in such cases shells of equal density turn as a whole. He proved also that the angular velocity in every case diminishes outwards. This paper is devoted to the study of the same problem with a somewhat different approach. It contains a complete integration of the problem and obtains the general expressions for the potential, the angular velocity and the pressure. The analysis also serves to bring out certain characteristic features in addition to obtaining the results arrived at by Dive.

The paper is divided into two sections. Sec. I contains the deduction of the general form for the potential function in elliptic coordinates and obtains the same in two particular cases, with a view to utilisation in part (II) of this study.

Sec. 2. contains a discussion of the general problem. At the end a few general formulae for some of the physical quantities connected with such bodies have also been appended.

Sec. 1. POTENTIALS OF BODIES IN CONFOCAL STRATIFICATIONS

1. *Confocal Ellipsoidal Stratifications.* We know that the potential at an internal point (x', y', z') of a homogeneous solid ellipsoid bounded by

$$\frac{x^2}{a^2-v} + \frac{y^2}{b^2-v} + \frac{z^2}{c^2-v} = 1 \quad (1.1)$$

is

$$U \equiv \pi\rho\{(a^2-v)(b^2-v)(c^2-v)\}^{\frac{1}{2}} \int_0^\infty \frac{d\theta}{\{(a^2-v+\theta)(b^2-v+\theta)(c^2-v+\theta)\}^{\frac{1}{2}}} \left\{1 - \sum \frac{x'^2}{a^2-v+\theta}\right\}. \quad (1.2)$$

Setting

$$A(v) = \int_0^\infty \frac{\{(a^2-v)(b^2-v)(c^2-v)\}^{\frac{1}{2}} d\theta}{\{(a^2-v+\theta)(b^2-v+\theta)(c^2-v+\theta)\}^{\frac{1}{2}}}$$

$$B(v) = \int_0^\infty \frac{\{(a^2-v)(b^2-v)(c^2-v)\}^{\frac{1}{2}} d\theta}{(a^2-v+\theta)^{3/2}(b^2-v+\theta)^{\frac{1}{2}}(c^2-v+\theta)^{\frac{1}{2}}}$$

and two similar expressions for $C(v)$ and $D(v)$, (1.2) can be put in the form

$$U = \pi\rho[A(v) - x'^2 B(v) - y'^2 C(v) - z'^2 D(v)]. \quad (1.8)$$

The potential of the elementary homogeneous shell of density ρ bounded outside by the ellipsoid (1.1) is

$$-\pi\rho \frac{d}{dv} [A(v) - x'^2 B(v) - y'^2 C(v) - z'^2 D(v)] dv.$$

If the boundary of the whole body be of semi-axes a, b, c and the confocal through (x', y', z') be

$$\frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} + \frac{z^2}{c^2 - \lambda} = 1 \quad (1.4)$$

we have, by summing up the contributions due to all the shells to which (x', y', z') is internal and considering the density for each shell to be a function $\rho(v)$ of its parameter v ,

$$U_i = \pi \int_{\lambda}^0 \rho(v) \frac{d}{dv} [A(v) - x'^2 B(v) - y'^2 C(v) - z'^2 D(v)] dv \quad (1.5)$$

as the potential due to all matter of the ellipsoid (a, b, c) outside the confocal through (x', y', z') .

Again, for the homogeneous solid ellipsoid

$$\frac{x^2}{a^2 - v'} + \frac{y^2}{b^2 - v'} + \frac{z^2}{c^2 - v'} = 1, \quad (1.6)$$

the potential, U_e , at an external point (x', y', z') is given by

$$U_e = \pi\rho \{(a^2 - v')(b^2 - v')(c^2 - v')\}^{\frac{1}{2}} \int_{\lambda_1}^{\infty} \frac{d\theta}{\{(a^2 - v' + \theta)(b^2 - v' + \theta)(c^2 - v' + \theta)\}^{\frac{1}{2}}} \left\{ 1 - \sum \frac{x'^2}{a^2 - v' + \theta} \right\} \quad (1.7)$$

where the confocal through (x', y', z') is given by

$$\frac{x^2}{a^2 - v' + \lambda_1} + \frac{y^2}{b^2 - v' + \lambda_1} + \frac{z^2}{c^2 - v' + \lambda_1} = 1. \quad (1.8)$$

As (1.4) and (1.8) represent the same ellipsoid, we have,

$$\lambda_1 = v' - \lambda. \quad (1.9)$$

With the substitution $-v' + \theta = \theta'$ (1.7) becomes

$$U_e = \pi\rho \{(a^2 - v')(b^2 - v')(c^2 - v')\}^{\frac{1}{2}} \int_{-\lambda}^{\infty} \frac{d\theta}{\{(a^2 + \theta)(b^2 + \theta)(c^2 + \theta)\}^{\frac{1}{2}}} \left\{ 1 - \sum \frac{x'^2}{a^2 + \theta} \right\}. \quad (1.10)$$

Putting

$$I = \int_{-\lambda}^{\infty} \frac{d\theta}{\{(a^2 + \theta)(b^2 + \theta)(c^2 + \theta)\}^{\frac{1}{2}}} \left\{ 1 - \sum \frac{x'^2}{a^2 + \theta} \right\} \quad (1.11)$$

we find that I is independent of v' and we have

$$U_e = \pi\rho \{(a^2 - v')(b^2 - v')(c^2 - v')\}^{\frac{1}{2}} I.$$

Now, the contribution of the shell of density $\rho(v')$ and with the ellipsoidal surface (1.6) as its outer boundary is

$$-\pi\rho I \frac{d}{dv} \{(a^2-v)(b^2-v)(c^2-v)\}^{\frac{1}{2}} dv.$$

Hence, the potential due to all the shells to which (x', y', z') is an external point is given by

$$U_s \equiv -\pi I \int_{\lambda}^{\infty} (v) \frac{d}{dv} \{(a^2-v)(b^2-v)(c^2-v)\}^{\frac{1}{2}} dv. \quad (1.12)$$

Now adding (1.5) and (1.12), the potential at the internal point (x', y', z') due to the whole ellipsoid (a, b, c) is given by

$$V_i = \pi \left[\int_{\lambda}^0 \rho(v) \frac{d}{dv} [A(v) - x'^2 B(v) - y'^2 C(v) - z'^2 D(v)] dv + I \int_0^{\lambda} \rho(v) \frac{d}{dv} \{(a^2-v)(b^2-v)(c^2-v)\}^{\frac{1}{2}} dv \right]. \quad (1.13)$$

It may be noted that in all the results given above we have taken the gravitational constant G to be unity.

2. Confocal Spheroidal Stratifications. In the case of a heterogeneous mass with stratifications in confocal spheroids and having the axis of z as the common axis of symmetry, the potential at the internal point whose cylindrical co-ordinates are (ξ, η) is given by

$$\frac{V_i}{\pi} = \int_{\lambda}^0 \rho(v) \frac{d}{dv} [A(v) - \xi^2 B(v) - \eta^2 C(v)] dv + I \int_{\lambda}^{\infty} \rho(v) \frac{d}{dv} \{(a^2-v)(c^2-v)\}^{\frac{1}{2}} dv \quad (1.14)$$

where the boundary is given by the meridian section

$$\frac{r^2}{a^2} + \frac{z^2}{c^2} = 1, \quad (1.15)$$

the density on the surface

$$\frac{r^2}{a^2-v} + \frac{z^2}{c^2-v} = 1 \quad (1.16)$$

is $\rho(v)$, λ is the positive root of the equation

$$\frac{\xi^2}{a^2-\lambda} + \frac{\eta^2}{c^2-\lambda} = 1, \quad (1.17)$$

and $A(v)$, $B(v)$, $C(v)$ and I are given by

$$\left. \begin{aligned} A(v) &= \int_0^{\infty} \frac{(a^2-v)(c^2-v)^{\frac{1}{2}} d\theta}{(a^2-v+\theta)(c^2-v+\theta)^{\frac{1}{2}}} \\ B(v) &= \int_0^{\infty} \frac{(a^2-v)(c^2-v)^{\frac{1}{2}} d\theta}{(a^2-v+\theta)^2(c^2-v+\theta)^{\frac{1}{2}}} \\ C(v) &= \int_0^{\infty} \frac{(a^2-v)(c^2-v)^{\frac{1}{2}} d\theta}{(a^2-v+\theta)(c^2-v+\theta)^{\frac{3}{2}}} \end{aligned} \right\} \quad (1.18)$$

$$I = \int_{-\lambda}^{\infty} \frac{d\theta}{(a^2 + \theta)(c^2 + \theta)^{\frac{1}{2}}} \left\{ 1 - \frac{\xi^2}{a^2 + \theta} - \frac{\eta^2}{c^2 + \theta} \right\}. \quad (1.19)$$

Now, putting

$$A(v) = f(v).L(v); \quad B(v) = f(v).M(v); \quad C(v) = f(v).N(v) \quad (1.20)$$

where

$$\left. \begin{aligned} f(v) &= (a^2 - v)(c^2 - v)^{\frac{1}{2}}, \\ L(v) &= \int_0^{\infty} \frac{d\theta}{(a^2 - v + \theta)(c^2 - v + \theta)^{\frac{1}{2}}}, \\ M(v) &= \int_0^{\infty} \frac{d\theta}{(a^2 - v + \theta)^2(c^2 - v + \theta)^{\frac{1}{2}}}, \\ N(v) &= \int_0^{\infty} \frac{d\theta}{(a^2 - v + \theta)(c^2 - v + \theta)^{\frac{3}{2}}}, \end{aligned} \right\} \quad (1.21)$$

and using the substitution $\theta = -\lambda + \theta'$, (1.19) reduces to

$$I = \int_0^{\infty} \frac{d\theta'}{(a^2 - \lambda + \theta')(c^2 - \lambda + \theta')^{\frac{1}{2}}} \left\{ 1 - \frac{\xi^2}{a^2 - \lambda + \theta'} - \frac{\eta^2}{c^2 - \lambda + \theta'} \right\}$$

and hence from (1.21) we have

$$I = L(\lambda) - \xi^2 M(\lambda) - \eta^2 N(\lambda). \quad (1.22)$$

Hence using (1.20), (1.18) and (1.22), (1.14) becomes

$$\begin{aligned} \frac{V_1}{\pi} &= \int_{\lambda}^0 \rho(v) \cdot \frac{d}{dv} [f(v) \{L(v) - \xi^2 M(v) - \eta^2 N(v)\}] dv \\ &\quad + [L(\lambda) - \xi^2 M(\lambda) - \eta^2 N(\lambda)] \int_0^{\lambda} \rho(v) \cdot \frac{df(v)}{dv} dv. \end{aligned} \quad (1.23)$$

which gives the potential of a heterogeneous spheroid with stratifications confocal to the boundary.

3. *Transformation to elliptic coordinates.* Let us put

$$\left. \begin{aligned} r &= k(1 - \mu^2)^{\frac{1}{2}}(1 + \zeta^2)^{\frac{1}{2}}, \\ z &= k\mu\zeta \end{aligned} \right\} \quad (1.24)$$

so that the curves $\zeta = \text{constant}$ and $\mu = \text{constant}$ are respectively the confocal ellipses and hyperbolas

$$\frac{r^2}{1 + \zeta^2} + \frac{z^2}{\zeta^2} = k^2 \quad (1.25)$$

and

$$\frac{r^2}{1 - \mu^2} - \frac{z^2}{\mu^2} = k^2, \quad (1.26)$$

both systems being confocal with the foci at $(\pm k, 0)$. We further assume that

$$(i) \quad \zeta = \zeta_1 \text{ corresponds to the boundary, } \frac{r^2}{a^2} + \frac{z^2}{c^2} = 1,$$

(ii) $\zeta = \zeta_0$, to the confocal through (ξ, η)

and

(iii) $\zeta = \zeta_1$, to the surface $\frac{r^2}{a^2 - v} + \frac{z^2}{c^2 - v} = 1$.

And so

$$\left. \begin{aligned} a^2 &= k^2(1 + \zeta_1^2) \\ c^2 &= k^2\zeta_1^2 \end{aligned} \right\}; \quad \left. \begin{aligned} a^2 - \lambda &= k^2(1 + \zeta_0^2) \\ c^2 - \lambda &= k^2\zeta_0^2 \end{aligned} \right\}; \quad \left. \begin{aligned} a^2 - v &= k^2(1 + \zeta^2) \\ c^2 - v &= k^2\zeta^2 \end{aligned} \right\} \quad (1.27)$$

hence $v = 0, \lambda, c^2$, correspond to $\zeta = \zeta_1, \zeta_0, 0$, respectively. (1.27a)

With (1.27) we have from (1.21)

$$f(v) = k^3\zeta(1 + \zeta^2); \quad L(v) = \frac{1}{k} L_1(\zeta); \quad M(v) = \frac{1}{k^3} M_1(\zeta); \quad N(v) = \frac{1}{k^3} N_1(\zeta),$$

where

$$\begin{aligned} L_1(\zeta) &= \int_0^\infty \frac{d\theta}{(1 + \zeta^2 + \theta)(\zeta^2 + \theta)^{\frac{1}{2}}}; \quad M_1(\zeta) = \int_0^\infty \frac{d\theta}{(1 + \zeta^2 + \theta)^2(\zeta^2 + \theta)^{\frac{1}{2}}}; \\ N_1(\zeta) &= \int_0^\infty \frac{d\theta}{(1 + \zeta^2 + \theta)(\zeta^2 + \theta)^{3/2}}. \end{aligned} \quad (1.28)$$

Now putting

$$A_1(\zeta) = k^2\zeta(1 + \zeta^2)L_1(\zeta), \quad B_1(\zeta) = \zeta(1 + \zeta^2)M_1(\zeta), \quad C_1(\zeta) = \zeta(1 + \zeta^2)N_1(\zeta) \quad (1.29)$$

(1.28) reduces to

$$\begin{aligned} \frac{V_1}{\pi} &= \int_{\zeta_1}^{\zeta_0} \rho(\zeta) \cdot \frac{d}{d\zeta} [A_1(\zeta) - \xi^2 B_1(\zeta) - \eta^2 C_1(\zeta)] d\zeta + [k^2 L_1(\zeta_0) - \xi^2 M_1(\zeta_0) - \eta^2 N_1(\zeta_0)] \\ &\quad \times \int_0^{\zeta_0} \rho(\zeta) \cdot (1 + 3\zeta^2) d\zeta. \end{aligned} \quad (1.30)$$

The point (ξ, η) is here defined by

$$\xi = k(1 - \mu^2)^{\frac{1}{2}}(1 + \zeta_0^2)^{\frac{1}{2}}, \quad \eta = k\mu\zeta_0. \quad (1.31)$$

The functions $L_1(\zeta)$, $M_1(\zeta)$, etc. defined in (1.28) and (1.29) can all be evaluated in explicit terms and we have

$$L_1(\zeta) = \pi - 2 \tan^{-1}\zeta, \quad M_1(\zeta) = \frac{\pi}{2} - \tan^{-1}\zeta - \frac{\zeta}{1 + \zeta^2}, \quad N_1(\zeta) = 2 \tan^{-1}\zeta + \frac{2}{\zeta} - \pi. \quad (1.32)$$

$$A_1(\zeta) = k^2\zeta(1 + \zeta^2)\{\pi - 2 \tan^{-1}\zeta\},$$

$$B_1(\zeta) = \zeta(1 + \zeta^2)\left\{\frac{\pi}{2} - \tan^{-1}\zeta - \frac{\zeta}{1 + \zeta^2}\right\}, \quad (1.33)$$

$$C_1(\zeta) = \zeta(1 + \zeta^2)\left\{2 \tan^{-1}\zeta + \frac{2}{\zeta} - \pi\right\}.$$

4. The formula (1.30) can be reduced by partial integration to

$$\begin{aligned} \frac{V_1}{\pi} &= \rho(\xi_1)\xi_1(1+\xi_1^2)[k^2L_1(\xi_1)-\xi^2M_1(\xi_1)-\eta^2N_1(\xi)] \\ &\quad - \int_{\xi_1}^{\xi_0} \rho'(\xi)\xi(1+\xi^2)[k^2L_1(\xi)-\xi^2M_1(\xi)-\eta^2N_1(\xi)]d\xi \\ &\quad - [k^2L_1(\xi_0)-\xi^2M_1(\xi_0)-\eta^2N_1(\xi_0)] \int_0^{\xi_0} \rho'(\xi)\xi(1+\xi^2)d\xi. \quad (1.34) \end{aligned}$$

In the case where the *density vanishes on the surface* we have

$$\rho(\xi_1) = 0 \quad (1.35)$$

and hence (1.34) reduces to

$$\begin{aligned} \frac{V_1}{\pi} &= -[k^2L_1(\xi_0)-\xi^2M_1(\xi_0)-\eta^2N_1(\xi_0)] \int_0^{\xi_0} \rho'(\xi)\xi(1+\xi^2)d\xi \\ &\quad - \int_{\xi_1}^{\xi_0} [k^2L_1(\xi)-\xi^2M_1(\xi)-\eta^2N_1(\xi)]\rho'(\xi)\xi(1+\xi^2)d\xi. \quad (1.36) \end{aligned}$$

5. We shall now prove the following theorem :

Whenever the density of a system of confocal spheroidal stratifications diminishes outwards the potential on any stratification increases from the equator to the pole along a meridian section.

As the value of μ on any meridional section of a stratification increases from zero at the equator to 1 at the pole, it will be enough to show that

$$\frac{\partial V}{\partial \mu^2} > 0, \quad 0 < \mu < 1 \quad (1.36)$$

whenever $d\rho/d\xi < 0$.

From (1.34) it is clear on writing down the values of ξ and η from (1.31) that

$$\begin{aligned} \frac{1}{\pi k^2} \frac{\partial V_1}{\partial \mu^2} &= \rho(\xi_1)\xi_1(1+\xi_1^2)[(1+\xi_0^2)M_1(\xi_1)-\xi_0^2N_1(\xi_1)] \\ &\quad - \int_{\xi_1}^{\xi_0} \rho'(\xi)\xi(1+\xi^2)[(1+\xi_0^2)M_1(\xi)-\xi_0^2N_1(\xi)]d\xi \\ &\quad - [(1+\xi_0^2)M_1(\xi_0)-\xi_0^2N_1(\xi_0)] \int_0^{\xi_0} \rho'(\xi)\xi(1+\xi^2)d\xi. \quad (1.37) \end{aligned}$$

As $\rho'(\xi) < 0$, the right-hand-side in (1.37) will be positive if

$$(1+\xi_0^2)M_1(\xi)-\xi_0^2N_1(\xi) > 0,$$

for $\xi \geq \xi_0$.

Now, from (1.28) we have

$$(1+\xi_0^2)M_1(\xi)-\xi_0^2N_1(\xi) = \int_0^\infty \frac{(\xi^2-\xi_0^2+\theta)d\theta}{(1+\xi^2+\theta)^2(\xi^2+\theta)^{3/2}}.$$

The integral is obviously positive for $\xi \geq \xi_0$ and hence the result follows.

6. As an illustration of the general results given above we shall evaluate the potential for the law of density

$$\rho(\xi) = \rho_0 \left(1 - \frac{m\xi}{(1+\xi^2)^{\frac{1}{2}}} \right) \quad (1.38)$$

for $m > 1$ and assume

$$\rho(\xi_1) = 0, \quad (1.38a)$$

ξ_1 corresponding to the boundary ellipsoid (1.15). Thus (1.36) becomes applicable and we have

$$m = \frac{(1+\xi_1^2)^{\frac{1}{2}}}{\xi_1}. \quad (1.38c)$$

From (1.38),

$$\rho'(\xi) = -\frac{m\rho_0}{(1+\xi^2)^{\frac{3}{2}}} \quad (1.39)$$

and hence (1.36) turns out to be

$$\begin{aligned} \frac{V_1}{\pi\rho_0 m} = & k^2 [\chi_1(\xi_1) - \{\pi - 2 \tan^{-1} \xi_0 + 2 \log (\xi_0 + (1+\xi_0^2)^{\frac{1}{2}})\}] \\ & - \xi^2 \left[\chi_2(\xi_1) - \left\{ \frac{\pi}{2} - \tan^{-1} \xi_0 - \frac{\xi_0}{1+\xi_0^2} + \frac{2\xi_0}{(1+\xi_0^2)^{\frac{1}{2}}} \right\} \right] \\ & - \eta^2 \left[-\chi_3(\xi_1) - \left\{ \frac{2}{\xi_0} + 2 \tan^{-1} \xi_0 - \pi - \frac{2(1+\xi_0^2)^{\frac{1}{2}}}{\xi_0} \right\} \right] \end{aligned} \quad (1.40)$$

where χ_1, χ_2, χ_3 are given by

$$\chi_1(\xi_1) = \pi(1+\xi_1^2)^{\frac{1}{2}} - 2 \tan^{-1} \xi_1(1+\xi_1^2)^{\frac{1}{2}} + 2 \log [\xi_1 + (1+\xi_1^2)^{\frac{1}{2}}] \quad (1.41a)$$

$$\chi_2(\xi_1) = \left(\frac{1}{2}\pi - \tan^{-1} \xi_1 \right) (1+\xi_1^2)^{\frac{1}{2}} + \frac{\xi_1}{(1+\xi_1^2)^{\frac{1}{2}}} \quad (1.41b)$$

$$\chi_3(\xi_1) = (\pi - 2 \tan^{-1} \xi_1) (1+\xi_1^2)^{\frac{1}{2}} \quad (1.41c)$$

at the point (ξ_0, μ) .

Now changing the coordinates of the point considered from (ξ_0, μ) to (ξ, μ) , we have, as the potential at the point (ξ, μ) ,

$$\begin{aligned} \frac{V_1(\xi, \mu)}{\pi\rho_0 m k^2} = & \{(\chi_1 - \pi) + 2 \tan^{-1} \xi - 2 \log [\xi + (1+\xi^2)^{\frac{1}{2}}]\} \\ & + (1-\mu^2)(1+\xi^2) \left\{ \left(\frac{1}{2}\pi - \chi_2 \right) - \tan^{-1} \xi - \frac{\xi}{1+\xi^2} + \frac{2\xi}{(1+\xi^2)^{\frac{1}{2}}} \right\} \\ & + \mu^2 \xi^2 \left\{ (\chi_3 - \pi) + 2 \tan^{-1} \xi + \frac{2}{\xi} - \frac{2(1+\xi^2)^{\frac{1}{2}}}{\xi} \right\} \end{aligned} \quad (1.42)$$

7. As a second application of the formula (1.36), the potential for the law of density

$$\rho = \rho_0(\xi_1 - \xi) \quad (1.43)$$

at the point (ξ, μ) is found to be

$$V_1(\zeta, \mu)/\pi\rho_0 \equiv k^2\{\psi_1(\zeta_1) + F_1(\zeta)\} - \xi^2\{\psi_2(\zeta_1) + F_2(\zeta)\} - \eta^2\{\psi_3(\zeta_1) + F_3(\zeta)\} \quad (1.44)$$

where (ξ, η) are given by (1.31) and

$$\psi_1(\zeta_1) = \frac{1}{2}(1 + \zeta_1^2)^2\left\{\frac{1}{2}\pi - \tan^{-1}\zeta_1\right\} + \frac{1}{2}\zeta_1(1 + \frac{1}{3}\zeta_1^2) \quad (1.45a)$$

$$\psi_2(\zeta_1) = \frac{1}{4}(1 + \zeta_1^2)^2\left\{\frac{1}{2}\pi - \tan^{-1}\zeta_1\right\} - \frac{1}{4}\zeta_1(1 + \frac{2}{3}\zeta_1^2) \quad (1.45b)$$

$$\psi_3(\zeta_1) = \frac{1}{2}(1 + \zeta_1^2)^2\left\{\tan^{-1}\zeta_1 - \frac{1}{2}\pi\right\} + \frac{3}{2}\zeta_1(1 + \frac{1}{3}\zeta_1^2) \quad (1.45c)$$

$$F_1(\zeta) = -\frac{1}{2}\left\{\frac{1}{2}\pi - \tan^{-1}\zeta + \zeta(1 + \frac{1}{3}\zeta^2)\right\} \quad (1.46a)$$

$$F_2(\zeta) = -\frac{1}{4}\left\{\frac{1}{2}\pi - \tan^{-1}\zeta - \frac{\zeta}{1 + \zeta^2} - \frac{2}{3}\zeta^3\right\} \quad (1.46b)$$

$$F_3(\zeta) = -\left\{-\frac{1}{2}\pi + \tan^{-1}\zeta + \frac{1}{2}\zeta - \frac{1}{3}\zeta^3\right\} \quad (1.46c)$$

Sec. 2. Equilibrium of Fluid-Bodies in confocal Stratifications

1. The equations of motion of a mass of perfect fluid rotating steadily about the axis of z , in cylindrical coordinates, are

$$\frac{\omega^2}{2} = -\frac{\partial\Phi}{\partial r^2} + \frac{1}{\rho} \frac{\partial p}{\partial r^2} \quad (2.1)$$

$$0 = -\frac{\partial\Phi}{\partial z^2} + \frac{1}{\rho} \frac{\partial p}{\partial z^2} \quad (2.2)$$

where ω, Φ, ρ, p stand for the angular velocity, the gravitational potential, the density and the pressure respectively.

We introduce elliptic coordinates by the substitutions (1.24), so that the curves $\zeta = \text{constant}$ are confocal ellipses and the curves $\mu = \text{constant}$ are confocal hyperbolas.

The equations (2.1) and (2.2) now reduce to

$$\frac{\partial\Phi}{\partial\zeta^2} = \frac{1}{\rho} \frac{\partial p}{\partial\zeta^2} - \frac{\omega^2 k^2}{2} (1 - \mu^2) \quad (2.3)$$

$$\frac{\partial\Phi}{\partial\mu^2} = \frac{1}{\rho} \frac{\partial p}{\partial\mu^2} + \frac{\omega^2 k^2}{2} (1 + \zeta^2), \quad (2.4)$$

and eliminating p from (2.3) and (2.4) we have

$$(1 + \zeta^2) \frac{\partial}{\partial\zeta^2} \left(\frac{\rho\omega^2}{2} \right) + (1 - \mu^2) \frac{\partial}{\partial\mu^2} \left(\frac{\rho\omega^2}{2} \right) = \frac{1}{k^2} \left\{ \frac{\partial\rho}{\partial\zeta^2} \cdot \frac{\partial\Phi}{\partial\mu^2} - \frac{\partial\rho}{\partial\mu^2} \cdot \frac{\partial\Phi}{\partial\zeta^2} \right\} \quad (2.5)$$

2. We now assume the boundary to be given by

$$\frac{r^2}{a^2} + \frac{z^2}{c^2} = 1 \quad (2.6)$$

as its meridian section, and also that on this surface $\zeta = \zeta_1$. Hence (2.6) must be the same as

$$\frac{r^2}{1 + \zeta_1^2} + \frac{z^2}{\zeta_1^2} = k^2 \quad (2.7)$$

so that

$$a^2 = k^2(1 + \zeta_1^2); \quad c^2 = k^2\zeta_1^2. \quad (2.8)$$

The equations imply that the eccentricity of the boundary is given by

$$e^2 = 1/(1 + \xi_1^2). \quad (2.9)$$

For stratifications confocal with the boundary we must have

$$\rho \equiv \rho(\xi). \quad (2.10)$$

The equation (2.5) now reduces to

$$(1 + \xi^2) \frac{\partial}{\partial \xi^2} \left(\frac{\rho \omega^2 k^2}{2} \right) + (1 - \mu^2) \frac{\partial}{\partial \mu^2} \left(\frac{\rho \omega^2 k^2}{2} \right) = \frac{d\rho}{d\xi^2} \cdot \frac{\partial \Phi}{\partial \mu^2}. \quad (2.11)$$

3. We shall now suppose that the field-potential Φ is due purely to *self-gravitation* of the body; hence Φ must be the same as V_i of the equation (1.30). Hence in this case we have, in general, at the point (ξ_0, μ)

$$\Phi(\xi_0, \mu) = V_i, \quad (2.12)$$

where V_i is given by (1.30) and where (ξ, η) are the cylindrical co-ordinates of the point (ξ_0, μ) and constants for the purposes of the integration in (1.30).

Now, it is well-known that in all problems of rotating fluids one of the four quantities mentioned in Art. 1 must be prescribed before the problem can be solved. We shall assume, in what follows, that the density of the model is prescribed with the boundary defined by (2.6) or (2.7). For a complete solution one has now to obtain, by integration, the angular velocity and the pressure. The validity of the solution is subject to the restriction that the square of the angular velocity, thus obtained, must be positive everywhere inside the body, and the pressure should either vanish on the surface or have a positive constant value everywhere on it. This latter condition will automatically ensure that the pressure everywhere inside is positive by virtue of equation (2.2), an argument essentially due to Dive.

4. *The angular velocity* Equation (2.11) being a linear partial differential equation of the first order can always be integrated and the allied differential equations are

$$\frac{d\xi^2}{1 + \xi^2} = \frac{d\mu^2}{1 - \mu^2} = d\left(\frac{1}{2}\rho\omega^2 k^2\right) / \left(\frac{\partial \Phi}{\partial \mu^2} \frac{d\rho}{d\xi^2}\right). \quad (2.13)$$

One integral of this equation is obviously

$$(1 + \xi^2)(1 - \mu^2) = \text{constant} \quad (2.14)$$

that is,

$$r^2 = \text{constant}. \quad (2.14a)$$

Another independent integral of (2.13) can, however, be obtained for all laws of the density stratifications. This is because $\partial \Phi / \partial \mu^2$ can be shown easily, from (2.12) to be a function of ξ alone. In fact, in Sec 1, equation (1.37), we have calculated $\partial \Phi / \partial \mu^2$ generally and it is obvious that the same is independent of μ . Hence, as a second independent integral of (2.13), we have

$$\frac{1}{2}\rho\omega^2 k^2 = \int \frac{1}{1 + \xi^2} \cdot \frac{\partial \Phi}{\partial \mu^2} \cdot \frac{d\rho}{d\xi^2} \cdot d\xi^2. \quad (2.15)$$

Thus, the general solution of (2.13) can be written as

$$\frac{1}{2}\rho\omega^2k^2 = \int \frac{1}{1+\zeta^2} \cdot \frac{\partial\Phi}{\partial\mu^2} \cdot \frac{d\rho}{d\zeta} \cdot d\zeta + F\{(1-\mu^2)(1+\zeta^2)\} \quad (2.16)$$

where F stands for any arbitrary function. Putting

$$\Omega(\zeta) \equiv \int \frac{1}{1+\zeta^2} \cdot \frac{\partial\Phi}{\partial\mu^2} \cdot \frac{d\rho}{d\zeta} \cdot d\zeta \quad (2.17)$$

and

$$F\{(1-\mu^2)(1+\zeta^2)\} \equiv \sum_{s=0}^{\infty} B_s \{(1-\mu^2)(1+\zeta^2)\}^s \quad (2.18)$$

we have

$$\frac{1}{2}\rho\omega^2k^2 = \Omega(\zeta) + \sum_{s=0}^{\infty} B_s \{(1-\mu^2)(1+\zeta^2)\}^s \quad (2.19)$$

5. *The Pressure.* From (2.3) and (2.4) we have at once

$$dp = \rho d\Phi + \frac{1}{2}\rho\omega^2k^2 \cdot d\{(1-\mu^2)(1+\zeta^2)\}. \quad (2.20)$$

With (2.19), this can be written as

$$dp = d(\rho\Phi) + \sum_{s=0}^{\infty} B_s \{(1-\mu^2)(1+\zeta^2)\}^s \cdot d\{(1-\mu^2)(1+\zeta^2)\} + \Omega(\zeta) d\{(1-\mu^2)(1+\zeta^2)\} - \Phi d\rho. \quad (2.21)$$

Now, from (2.12) it is clear that we can put Φ in the form

$$\Phi(\zeta, \mu) = \Phi_1(\zeta) + \mu^2\Phi_2(\zeta), \quad (2.22)$$

so that, obviously

$$\frac{\partial\Phi}{\partial\mu^2} = \Phi_2(\zeta). \quad (2.23)$$

Now with (2.17) and (2.23), (2.21) can be put in the form

$$dp = d\left[\rho\Phi + \sum_{s=0}^{\infty} \frac{1}{s+1} \cdot B_s \{(1-\mu^2)(1+\zeta^2)\}^{s+1} - \Omega(\zeta)\mu^2(1+\zeta^2)\right] + \left[\Omega(\zeta) - \Phi_1(\zeta) \frac{d\rho}{d\zeta}\right] d\zeta^2. \quad (2.24)$$

This shows that for all laws of confocal stratifications dp is a perfect differential and hence the pressure can be integrated out without any further restrictions. We have

$$p = \rho\Phi - \mu^2(1+\zeta^2)\Omega(\zeta) + \sum_{s=0}^{\infty} \frac{1}{s+1} \cdot B_s \{(1-\mu^2)(1+\zeta^2)\}^{s+1} + \mathfrak{F}(\zeta) + C \quad (2.25)$$

where

$$\mathfrak{F}(\zeta) = \int \left\{ \Omega(\zeta) - \Phi_1(\zeta) \cdot \frac{d\rho}{d\zeta} \right\} d\zeta^2, \quad (2.26)$$

and C is the constant of integration.

6. *The Boundary Conditions* As we have already mentioned, the boundary condition is that the pressure either vanishes on the surface $\zeta = \zeta_1$ or is constant over it. In what follows we shall formulate our arguments on the assumption that the pressure vanishes on the surface. This will not make any difference in our general conclusions for models under constant pressure at the boundary. A reflection on the formula (2.25)

together with the equation (2.22), shows quite clearly that the pressure *cannot* vanish everywhere on the surface $\zeta = \zeta_1$, unless the terms in μ^4 and higher powers of μ^2 , all vanish. This means that the only non-zero term that can exist in the summation (2.18) is B_0 . Thus the constancy of the pressure or its vanishing, over the boundary determines exactly the form of the arbitrary function that appears in the expression for ω^2 (Eqn. 2.19). We now have (2.19) and (2.25) as,

$$\frac{1}{2}\rho\omega^2k^2 = \Omega(\zeta) + B_0 \quad (2.27)$$

and

$$p = \rho\Phi - \mu^2(1 + \zeta^2)\Omega(\zeta) + B_0(1 - \mu^2)(1 + \zeta^2) + \mathfrak{F}(\zeta) + C. \quad (2.28)$$

Now, from (2.17), (2.28) and (2.26) we have

$$\mathfrak{F}(\zeta) = (1 + \zeta^2)\Omega(\zeta) - \int (\Phi_1 + \Phi_2) \frac{d\rho}{d\zeta} d\zeta. \quad (2.29)$$

Hence, with (2.27), (2.28) reduces to,

$$p = \rho\Phi + (1 - \mu^2)(1 + \zeta^2)\frac{1}{2}\rho\omega^2k^2 - \int (\Phi_1 + \Phi_2) \frac{d\rho}{d\zeta} d\zeta + C. \quad (2.30)$$

Hence splitting up Φ as in (2.22) and equating to zero the term independent of μ^2 and the coefficient of μ^2 , separately, we have, as the conditions for the vanishing of the pressure over the boundary,

$$\rho\{\Phi_1 + \frac{1}{2}(1 + \zeta^2)\omega^2k^2\} - \int (\Phi_1 + \Phi_2) \frac{d\rho}{d\zeta} d\zeta + C = 0 \quad (2.31)$$

and

$$\rho\{\Phi_2 - \frac{1}{2}(1 + \zeta^2)\omega^2k^2\} = 0 \quad (2.32)$$

on $\zeta = \zeta_1$. Equations (2.31) and (2.32) determine the constant C and the boundary value, ω_s , of ω , in general. From (2.32) we find that when

$$(i) \quad \rho(\zeta_1) \neq 0, \quad \omega_s^2 = \frac{2}{k^2} \cdot \frac{1}{1 + \zeta_1^2} \Phi_2(\zeta_1) \quad (2.33)$$

but when (ii) $\rho(\zeta_1) = 0$, ω_s is not given by (2.32). We shall see soon how ω_s can be determined when the density vanishes on the surface.

In either case, however, C is obtained definitely from (2.31).

7. We noticed that the problems under discussion fall into two classes according as the density vanishes on the surface, or it does not.

(i) *Class (a):* $\rho(\zeta_1) = 0$. In this case, we have from (2.27)

$$\frac{1}{2}\rho\omega^2k^2 = \Omega(\zeta) - \Omega(\zeta_1) \quad (2.34)$$

and

$$p = \rho\Phi + \frac{1}{2}\rho\omega^2k^2(1 - \mu^2)(1 + \zeta^2) + \int_{\zeta_1}^{\zeta} (\Phi_1 + \Phi_2) \frac{d\rho}{d\zeta} d\zeta \quad (2.35)$$

and from (2.34) we find

$$\omega_s^2 = \frac{2}{k^2} \lim_{\zeta \rightarrow \zeta_1} \frac{\Omega(\zeta) - \Omega(\zeta_1)}{\rho(\zeta)} \quad (2.36)$$

ω_s being as before the surface value of ω .

(ii) *Class (b).* $\rho(\zeta_1) \neq 0$: Here (2.27) reduces to

$$\frac{1}{2}\rho\omega^2k^2 = \frac{1}{2}\rho(\zeta_1)\omega_s^2k^2 + \Omega(\zeta) - \Omega(\zeta_1) \quad (2.37)$$

where ω_s^2 is given by (2.38), and the pressure is given by

$$p = \rho\Phi + (1 - \mu^2)(1 + \zeta^2) \cdot \frac{1}{2}\rho\omega^2k^2 + \int_{\zeta}^{\zeta_1} (\Phi_1 + \Phi_2) \frac{d\rho}{d\zeta} d\zeta - \rho(\zeta_1) \cdot \{\Phi_1 + \Phi_2\}k^2 \quad (2.38)$$

which is obtained from (2.30) by substituting the value of C from (2.31).

8. It now remains to show that ω^2 given by (2.34), or by (2.37) is positive for every $\zeta \leq \zeta_1$. For this, we notice from (2.17) that

$$\frac{d}{d\zeta} \Omega(\zeta) = \frac{1}{1 + \zeta^2} \frac{\partial \Phi}{\partial \mu^2} \frac{d\rho}{d\zeta},$$

and it has been proved in Art 5, Sec 1 that $\partial \Phi / \partial \mu^2$ must everywhere be positive. Hence if the density diminishes outwards the function $\Omega(\zeta)$ must also behave in the same manner. Thus for all laws of density for which the density diminishes outwards we must have

$$d\Omega/d\zeta < 0. \quad (2.39)$$

Now ζ increases from zero at the central disc to ζ_1 at the outer boundary and $\Omega(\zeta)$ diminishes outwards. Hence

$$\Omega(\zeta_1) < \Omega(\zeta).$$

Thus we have

$$\Omega(\zeta) - \Omega(\zeta_1) > 0 \quad (2.40)$$

whenever $d\rho(\zeta)/d\zeta < 0$. Besides, by (2.28), $\omega_s^2 > 0$ because $\Phi_2(\zeta_1) > 0$, from (2.33).

This establishes completely that ω^2 obtained from (2.34) or (2.37) must be positive everywhere inside whenever the density of the stratifications diminishes monotonically outwards.

9. We shall now show that in either of the two cases the angular velocity must diminish outwards. From (2.37) we have

$$\frac{d}{d\zeta} (\frac{1}{2}\rho\omega^2k^2) = \Omega'(\zeta).$$

Hence, from (2.17), we have

$$\rho^2 \frac{d}{d\zeta} (\frac{1}{2}\omega^2k^2) = \frac{d\rho}{d\zeta} \left\{ \frac{\rho}{1 + \zeta^2} \frac{\partial \Phi}{\partial \mu^2} - \frac{\rho\omega^2k^2}{2} \right\}.$$

Now, substituting from (2.37) again, and simplifying, we have

$$\rho^2 \frac{d}{d\zeta} (\frac{1}{2}\omega^2k^2) = -\frac{d\rho}{d\zeta} \int_{\zeta}^{\zeta_1} \frac{d}{d\zeta} \left\{ \frac{1}{1 + \zeta^2} \frac{\partial \Phi}{\partial \mu^2} \right\} \rho \cdot d\zeta. \quad (2.41)$$

Now differentiating the expression for $\partial V(\zeta_0, \mu)/\partial \mu^2$ obtained in (1.37) with respect to ζ_0 and simplifying we have

$$\begin{aligned}
\frac{1}{\pi k^2} \cdot \frac{d}{d\zeta_0} \left\{ \frac{\partial \Phi(\zeta_0, \mu)}{\partial \mu^2} \right\} &= 2\zeta_0 [\rho(\zeta_1) \zeta_1 (1 + \zeta_1^2) \{M_1(\zeta_1) - N_1(\zeta_1)\} \\
&+ \int_{\zeta_0}^{\zeta_1} \{M_1(\zeta) - N_1(\zeta)\} \{-\rho'(\zeta)\} \zeta (1 + \zeta^2) d\zeta] \\
&+ \frac{d}{d\zeta_0} [(1 + \zeta_0^2) M_1(\zeta_0) - \zeta_0^2 N_1(\zeta_0)] \cdot \int_0^{\zeta_0} \{-\rho'(\zeta)\} \zeta (1 + \zeta^2) d\zeta. \quad (2.42)
\end{aligned}$$

Taking note of the fact that $\rho'(\zeta)$ is negative, we see that

- (1) The first term on the right hand side of (2.42) is negative if

$$M_1(\zeta_1) - N_1(\zeta_1) < 0; \quad (2.43a)$$

- (2) The 2nd term in the same expression is negative if

$$M_1(\zeta) - N_1(\zeta) < 0, \quad \text{for } \zeta_0 < \zeta < \zeta_1; \quad (2.43b)$$

- (3) The third term is negative if

$$\frac{d}{d\zeta_0} \{(1 + \zeta_0^2) M_1(\zeta_0) - \zeta_0^2 N_1(\zeta_0)\} < 0 \quad (2.43c)$$

Now from the values of $M_1(\zeta)$ and $N_1(\zeta)$ given in (1.28) it becomes clear that all the conditions mentioned in the equations (2.43a), (2.43b) and (2.43c) are satisfied. Hence from (2.42) we have

$$\frac{d}{d\zeta} \left(\frac{\partial \Phi}{\partial \mu^2} \right) < 0 \quad (2.44)$$

Thus, the integrand in (2.41) is negative and so the right-hand-side of equation (2.41) is negative whenever $d\rho/d\zeta < 0$. Hence

$$\frac{d}{d\zeta} \{ \frac{1}{2} \omega^2 k^2 \} < 0 \quad (2.45)$$

This proves that the angular velocity must diminish outwards. The result remains true when the density vanishes on the surface.

10. From the above analysis we conclude as follows:

(i) Whenever the stratifications are in spheroids confocal with the boundary, equilibrium under self-gravitation is possible for any law of variation of the density, provided the density continuously diminishes outwards.

(ii) In every such equilibrium the angular velocity is constant over the same stratification and hence the shells of equal density turn as a whole.

(iii) The angular velocity varies from shell to shell and continuously diminishes outwards under the same proviso as in (i)

These general conclusions were known. They were discovered by P. Dive (1930) in his researches on the internal rotations of stellar fluids. Dive obtained the results without the aid of the general formula for potentials of bodies in confocal stratifications, though, in his method he had practically integrated the force-components to the extent necessary for proving the reality of ω .

The present discussion is a somewhat different approach to the problem. It is to be noted that the success of the solution really depends on the characteristic form (2.22) of the potential function viz. $\Phi_1(\xi) + \mu^2 \Phi_2(\xi)$, in which $\Phi_1(\xi) > 0$. This is responsible for the interesting result (ii). Formulae (2.33) and (2.36) which give the surface angular velocity in terms of k and the surface eccentricity ($e^2 = 1/(1 + \xi_1^2)$) are however believed to be new.

Besides, a general expression for the pressure has been worked out.

It appears from the above discussion that if in addition to the field due to self-gravitation an external field is imposed which does not disturb the features that the total Φ can be put in the form $\Phi_1(\xi) + \mu^2 \Phi_2(\xi)$ and that $\Phi_1(\xi) > 0$, equilibrium in confocal stratifications may still be possible if $\partial\Phi/\partial z$ is negative everywhere, and then the angular velocity will maintain the same character.

11. It is possible to obtain in quite simple terms the general expressions for the mass, moment of inertia, etc. of models in confocal stratifications. As we shall have occasion to use several of them in our future work these expressions are given below in forms which may easily be integrated in particular cases.

(a) *The Mass.*

$$M = \int \rho dv = \int_0^{\xi_1} \rho \frac{d}{d\xi} \left\{ \frac{4}{3} \pi k^3 \xi (1 + \xi^2) \right\} d\xi = \frac{4}{3} \pi k^3 \left[\rho(\xi_1) \xi_1 (1 + \xi_1^2) - \int_0^{\xi_1} \rho'(\xi) \xi (1 + \xi^2) d\xi \right] \quad (2.46a)$$

in general. When the density vanishes on the surface we have

$$M = - \frac{4}{3} \pi k^3 \int_0^{\xi_1} \rho'(\xi) \xi (1 + \xi^2) d\xi. \quad (2.46b)$$

(b) *The kinetic energy.*

Putting

$$ds_1 = k \frac{(\mu^2 + \xi^2)^{\frac{1}{2}}}{(1 + \xi^2)^{\frac{1}{2}}} d\xi, \quad ds_2 = k \frac{(\mu^2 + \xi^2)^{\frac{1}{2}}}{(1 - \mu^2)^{\frac{1}{2}}} d\mu, \quad \text{we have}$$

$$K.E. = \frac{2\pi}{15} k^3 \int_0^{\xi_1} (1 + \xi^2)(1 + 5\xi^2) \rho \omega^2 k^2 d\xi \quad (2.47a)$$

When $\rho(\xi_1) = 0$, we have

$$K.E. = - \frac{4\pi}{15} k^3 \int_0^{\xi_1} \frac{\partial \phi}{\partial \mu^2} \cdot \frac{d\rho}{d\xi} \xi (1 + \xi^2) d\xi \quad (2.47b)$$

(c) *The moment of inertia, I about the axis of rotation.*

$$I = \frac{8\pi k^5}{15} \int_0^{\xi_1} \rho(\xi) \frac{d}{d\xi} \left\{ \xi (1 + \xi^2)^2 \right\} d\xi \quad (2.48a)$$

in general. And

$$I = - \frac{8\pi k^5}{15} \int_0^{\xi_1} \rho'(\xi) \xi (1 + \xi^2)^2 d\xi \quad (2.48b)$$

when $\rho(\xi_1) = 0$

(a) *The average of the squared angular-velocity.* As the average over volume of simple ω^2 leads to intractable integrals it has been found advantageous to calculate the average of the function $\rho \omega^2$. Two averages in this connection have been considered.

The first is the average over the values of the confocal parameter ζ and the second the is the average over the whole volume. Thus,

$$(i) \quad (\overline{\rho\omega^2})_{\text{shell}} = \frac{2}{k^2} \cdot \frac{1}{\zeta_1} \int_0^{\zeta_1} \{\Omega(\zeta) - \Omega(\zeta_1)\} d\zeta = -\frac{2}{k^2} \cdot \frac{1}{\zeta_1} \int_0^{\zeta_1} \frac{\partial\phi}{\partial\mu^2} \cdot \frac{d\rho}{d\zeta} \cdot \frac{\zeta}{1+\zeta^2} d\zeta. \quad (2.49a)$$

$$(ii) \quad (\overline{\rho\omega^2})_{\text{vol}} = -\frac{2}{k^2} \cdot \frac{1}{\zeta_1(1+\zeta_1^2)} \int_0^{\zeta_1} \frac{\partial\phi}{\partial\mu^2} \cdot \frac{d\rho}{d\zeta} d\zeta. \quad (2.49b)$$

Further, we define

$$(\overline{\omega^2})_{\text{mass}} = \int \omega^2 \rho dv / \int \rho dv = \frac{2}{k^2} \left[\int_0^{\zeta_1} \frac{\partial\phi}{\partial\mu^2} \cdot \frac{d\rho}{d\zeta} d\zeta / \int_0^{\zeta_1} \frac{d\rho}{d\zeta} \cdot \zeta \cdot (1+\zeta^2) d\zeta \right] \quad (2.50)$$

when $\rho(\zeta_1) = 0$

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EQUILIBRIUM OF ROTATING FLUID-BODIES IN CONFOCAL STRATIFICATIONS—II

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Introduction

In part I (Ghosh, 1950B) we have given a general discussion of the problem of rotating equilibrium of a spheroidal fluid-mass in confocal stratifications.

In the present paper we have worked out fully the case where the law of density is given by

$$\rho = \rho_0 \left(1 - \frac{m\zeta}{(1+\zeta^2)^{\frac{1}{2}}} \right), \quad m > 1 \quad (3.1)$$

ζ being the ellipsoidal co-ordinate used in (I). This case is shown to be completely integrable in closed form in terms of ordinary functions.

We have studied the 'average' value of the function $\omega^2/2\pi\rho$ for this model and the results show that qualitatively, it is of the same character as in the case of the homogeneous model or as in the case of the model where the distribution of density is in similar spheroids (Ghosh, 1950A). The variations in the average angular velocity, the central density and the equatorial extension with changing eccentricity have also been studied. The results show that such a model can lie only between the flat-disc state and a state of extreme condensation in the form of a droplet. It can never attain a state of infinite diffusion.

In the following discussion all equations under (1.) and (2.) refer to part I. Equations of this paper are marked as (3.).

2. Assuming as in (I) that ζ_1 stands for the boundary value of ζ and that

$$\rho(\zeta_1) = 0 \quad (3.2)$$

we have

$$m = (1 + \zeta_1^2)^{\frac{1}{2}} / \zeta_1. \quad (3.3)$$

In sec I of (I) we have shown that the potential due to self-gravitation of such a model is given by (1.43). Hence, for equilibrium under self gravitation only we have

$$\begin{aligned} \frac{\Phi}{\pi\rho_0 m k^2} = & \{(\chi_1 - \pi) + 2 \tan^{-1} \zeta - 2 \log [\zeta + (1 + \zeta^2)^{\frac{1}{2}}]\} + (1 - \mu^2)(1 + \zeta^2) \left\{ \left(\frac{1}{2}\pi - \chi_2 \right) - \tan^{-1} \zeta \right. \\ & \left. - \frac{\zeta}{1 + \zeta^2} + \frac{2\zeta}{(1 + \zeta^2)^{\frac{1}{2}}} \right\} + \mu^2 \zeta^2 \left\{ (\chi_3 - \pi) + 2 \tan^{-1} \zeta + \frac{2}{\zeta} - \frac{2(1 + \zeta^2)^{\frac{1}{2}}}{\zeta} \right\} \end{aligned} \quad (3.4)$$

where χ_1, χ_2, χ_3 are defined by the equations (1.41).

As the density of this model is supposed to vanish on the surface, ω and p should be given by equations (2.34) and (2.35), the surface value of ω^2 being determined by (2.36).

From (3.4) we find that in this case

$$\begin{aligned} \Phi_1/\pi\rho_0mk^2 = & \{(\chi_1 - \pi) + 2 \tan^{-1} \zeta - 2 \log [\zeta + (1 + \zeta^2)^{\frac{1}{2}}]\} \\ & + (1 + \zeta^2) \left\{ \left(\frac{1}{2}\pi - \chi_2 \right) - \tan^{-1} \zeta - \frac{\zeta}{1 + \zeta^2} + \frac{2\zeta}{(1 + \zeta^2)^{\frac{3}{2}}} \right\} \end{aligned} \quad (3.5)$$

$$\begin{aligned} \Phi_2/\pi\rho_0mk^2 = & \zeta^2 \left\{ (\chi_2 - \pi) + 2 \tan^{-1} \zeta + \frac{2}{\zeta} - \frac{2(1 + \zeta^2)^{\frac{1}{2}}}{\zeta} \right\} \\ & - (1 + \zeta^2) \left\{ \left(\frac{1}{2}\pi - \chi_2 \right) - \tan^{-1} \zeta - \frac{\zeta}{1 + \zeta^2} + \frac{2\zeta}{(1 + \zeta^2)^{\frac{3}{2}}} \right\}, \end{aligned} \quad (3.6)$$

so that $\Phi_1 + \Phi_2$ is given by the simpler form

$$\begin{aligned} \frac{\Phi_1 + \Phi_2}{\pi\rho_0mk^2} = & (\chi_1 - \pi) + 2 \tan^{-1} \zeta - 2 \log [\zeta + (1 + \zeta^2)^{\frac{1}{2}}] \\ & + \zeta^2 (\chi_1 - \pi) + 2\zeta^2 \tan^{-1} \zeta + 2\zeta - 2\zeta(1 + \zeta^2)^{\frac{1}{2}}. \end{aligned} \quad (3.7)$$

Hence substituting, we have from (2.17)

$$\Omega(\zeta) = -\pi\rho_0^2 m^2 k^2 W(\zeta) \quad (3.8)$$

where

$$\begin{aligned} W(\zeta) = & (\chi_2 - \frac{1}{2}\pi) \frac{\zeta}{(1 + \zeta^2)^{\frac{1}{2}}} + \frac{1}{2}(\chi_3 - \pi) \frac{\zeta^2}{(1 + \zeta^2)^{3/2}} \\ & + \frac{\zeta \tan^{-1} \zeta}{(1 + \zeta^2)^{\frac{1}{2}}} \left\{ 1 + \frac{2}{3} \frac{\zeta^2}{1 + \zeta^2} \right\} + \frac{5}{9} \frac{1}{(1 + \zeta^2)^{\frac{1}{2}}} - \frac{11}{9} \frac{1}{(1 + \zeta^2)^{3/2}} + \frac{2}{1 + \zeta^2}. \end{aligned} \quad (3.9)$$

Hence, from (2.34), we have

$$\rho\omega^2 = 2\pi\rho_0^2 m^2 [W(\zeta_1) - W(\zeta)] \quad (3.10)$$

and from (2.36) we obtain after some simplification

$$\begin{aligned} \frac{\omega^2}{2\pi\rho_0 m} = & \left[(\chi_2 - \frac{1}{2}\pi) + \frac{1}{2}(\chi_3 - \pi) \frac{\zeta_1^2}{1 + \zeta_1^2} - \zeta_1 \left\{ \frac{5}{9} - \frac{11}{9} \frac{1}{1 + \zeta_1^2} + 4 \frac{1}{(1 + \zeta_1^2)^{\frac{3}{2}}} \right\} + \zeta_1 + \tan^{-1} \zeta_1 \right. \\ & \left. + \{3 \tan^{-1} \zeta_1 + \zeta_1\} \frac{\zeta_1^2}{1 + \zeta_1^2} \right] \end{aligned} \quad (3.11)$$

This shows that the angular velocity at the surface does not vanish and it will be useful to remember that the angular velocity diminishes outwards in every case of confocal stratifications with density diminishing outwards.

From (2.35) we have by (3.7)

$$p = \rho\Omega^2 + \frac{1}{2}\rho\omega^2 k^2 (1 - \mu^2)(1 + \zeta^2) + 2\pi\rho_0^2 m^2 k^2 \Theta(\zeta) \quad (3.12)$$

where

$$\Theta(\zeta) = \left[\frac{\zeta}{1 + \zeta^2} \log \frac{\zeta_1 + (1 + \zeta_1^2)^{\frac{1}{2}}}{\zeta + (1 + \zeta^2)^{\frac{1}{2}}} - \frac{1}{(1 + \zeta^2)^{\frac{1}{2}}} + \frac{1}{2}(\chi_3 - \pi) \log \left\{ \zeta + (1 + \zeta^2)^{\frac{1}{2}} \right\} \right]_{\zeta_1}^{\zeta} + \int_{\zeta_1}^{\zeta} \frac{\tan^{-1} \zeta}{(1 + \zeta^2)^{\frac{1}{2}}} d\zeta \quad (3.13)$$

3 The values of the various physical quantities introduced in (I), [equations (2.46—2.50)] work out as follows in the present case.

(a) The mass is given by

$$M = \frac{4}{3}\pi k^3 \rho_0 m \{(1 + \zeta_1^2)^{\frac{1}{2}} - 1\} \quad (3.14a)$$

(b) The volume by

$$V = \frac{4}{3}\pi k^3 \zeta_1 (1 + \zeta_1^2) \quad (3.14b)$$

(c) The moment of inertia by

$$I = \frac{8\pi\rho_0 k^5}{45} \frac{(1 + \zeta_1^2)^{\frac{1}{2}}}{\zeta_1} \{(1 + \zeta_1^2)^{3/2} - 1\} \quad (3.14c)$$

(d) The kinetic energy by

$$K.E = \frac{4\pi^2 m^2 \rho_0^2 k^5}{15} \left[\frac{1}{3} \{1 + \zeta_1^2\}^{3/2} \{\chi_2 + \chi_3 - \frac{3}{2}\pi + 3 \tan^{-1} \zeta_1\} - \{\chi_2 + \chi_3 - \frac{3}{2}\pi\} + \{\zeta_1(1 + \zeta_1^2)^{\frac{1}{2}} - \frac{1}{3}\zeta_1^3\} \right] \quad (3.14d)$$

(e) The average density, $\bar{\rho}$ by

$$\bar{\rho} \text{ (vol)} = \rho_0 \frac{1}{\zeta_1^2} \left(1 - \frac{1}{(1 + \zeta_1^2)^{\frac{1}{2}}} \right) \quad (i)$$

$$\bar{\rho} \text{ (shell)} = \rho_0 \frac{1}{\zeta_1^3} [(1 + \zeta_1^2)^{\frac{1}{2}} - 1] \quad (ii) \quad (3.14e)$$

$$\bar{\rho} \text{ (mass)} = \frac{8m\rho_0^2 \pi k^3}{8M} \{m \tan^{-1} \zeta_1 - 1\} \quad (iii)$$

(f) The mass-average of the square of the angular velocity is,

$$\bar{\omega^2} = \int \omega^2 \cdot \rho dv / \int \rho dv \quad (3.14f)$$

where $v = \frac{4}{3}\pi k^3 \zeta_1 (1 + \zeta_1^2)$. From (3.14f) with (3.14e, (iii)) we have from (3.10)

$$\frac{\omega^2_{\text{mass}}}{\pi \rho_{\text{mass}}} = \left(\frac{\bar{\omega^2}}{\pi \rho} \right)_{\text{mass}} = \frac{P(\zeta_1)}{Q(\zeta_1)} \quad (3.15)$$

where

$$P(\zeta_1) = \left(\frac{1}{2}\pi - \tan^{-1} \zeta_1 \right) \left\{ 3(1 + \zeta_1^2) - 8(1 + \zeta_1^2)^{\frac{1}{2}} - \frac{2}{(1 + \zeta_1^2)^{\frac{1}{2}}} - 2 \right\} - \frac{6\zeta_1}{(1 + \zeta_1^2)^{\frac{1}{2}}} - 3\zeta_1 + \frac{9\pi}{2} \quad (3.16)$$

and

$$Q(\zeta_1) = \tan^{-1} \zeta_1 - \zeta_1 / (1 + \zeta_1^2)^{\frac{1}{2}}. \quad (3.17)$$

3. $\bar{\omega^2}/2\pi\bar{\rho}$ as a function of ζ_1 .

From (3.15) it can be shown that

$$L_t \frac{\bar{\omega^2}}{2\pi\bar{\rho}} = 0.119, \quad L_t \frac{\bar{\omega^2}}{2\pi\bar{\rho}} = 0. \quad (3.18)$$

The values of $\bar{\omega^2}/2\pi\bar{\rho}$ for a few other values of ζ_1 are shown in the table given below. Since ζ_1 corresponds to the eccentricity of the bounding spheroid of the model ($\zeta_1 \rightarrow \infty$ when $e \rightarrow 0$, and $\zeta_1 \rightarrow 0$ when $e \rightarrow 1$) we notice that in the spherical state the function attains a finite value and vanishes in the flat-disc state. In between it attains the maximum value 0.84 when $\zeta_1 = 0.4$, the corresponding value of e being about .92.

It is well-known that in the case of the homogeneous model (Maclaurin's Spheroid) the function vanishes at the two extreme limits and attains the maximum value 0.225 when the eccentricity is about 0.93.

In a previous paper (Ghosh, 1950A) we have shown how the *volume-average* of $\omega^2/2\pi\rho$ behaves for the law of distribution of density, $\rho = \rho_0(1 - \alpha r^2 - \beta z^2)$, in similar spheroids. In this case also the function vanishes at the two limits and attains the maximum 0.88 at $e = 0.92$.

For the sake of a better comparison with the latter case i.e., with $\rho = \rho_0(1 - \alpha r^2 - \beta z^2)$, we have calculated the mass average of the function $\omega^2/2\pi\rho$ for this case of similar distribution and the results are shown in the table. It is found that in the case of the mass average of the similar distribution, the function again vanishes at the two extremes but the maximum attained is only about 0.29 for nearly the same eccentricity 0.92.

Thus for the similar model the mass-average is nearer the homogeneous model than the volume average, but the qualitative behaviour of the two averages is practically the same.

Hence the mass-average introduced in the study of the present model will not lead us very far wrong, so far as the average behaviour of the function $\omega^2/2\pi\rho$ is concerned.

| ξ_1 | e | $\bar{\omega^2}/2\pi\bar{\rho}$ (mass) Present model | Similar model | |
|----------|-------|---|---------------------------------------|--|
| | | | $\bar{\omega^2}/2\pi\bar{\rho}$ (vol) | $\bar{\omega^2}/2\pi\bar{\rho}$ (mass) |
| ∞ | 0 | 0.119 | 0 | 0 |
| 10 | 0.09 | 0.148 | 0.0043 | — |
| 2 | 0.447 | 0.209 | 0.1227 | — |
| 1 | 0.707 | 0.254 | 0.235 | 0.1757 |
| 0.5 | 0.89 | 0.277 | 0.368 | 0.279 |
| 0.4 | 0.92 | 0.340 | 0.380 | 0.289 |
| 0.3 | 0.957 | 0.30 | 0.370 | 0.284 |
| 0 | 1 | 0 | 0 | 0 |

5. In the previous paper mentioned above we have investigated the way in which the central-density, the average angular-velocity and the equatorial extension of a fluid mass in similar stratifications change, as the model undergoes secular variations in the boundary eccentricity, keeping its mass and angular momentum constant. We propose now to undertake a similar study with respect to the present model, where the law of stratifications in confocal apheroids is given by (8.1).

Assuming the boundary to be

$$\frac{r^2}{a^2} + \frac{z^2}{c^2} = 1 \quad (8.19)$$

we must have

$$\left. \begin{aligned} a^2 &= k^2(1 + \xi_1^2) \\ c^2 &= k^2 \xi_1^2 \end{aligned} \right\} \quad (3.19a)$$

and the eccentricity of the boundary must be given by

$$e^2 = \frac{1}{1 + \xi_1^2} \quad (3.20)$$

which implies

$$\left. \begin{aligned} L_t \xi_1 &\rightarrow \infty, & e &\rightarrow 0 \\ L_t \xi_1 &\rightarrow 0, & e &\rightarrow 1 \end{aligned} \right\} \quad (3.21)$$

Now, the angular momentum of the model, given by

$$A = \int \omega r^2 \rho dv$$

cannot be evaluated easily. As we are interested only in qualitative results, we may without serious error, write

$$A = \bar{\omega} I \quad (3.22)$$

where the moment of inertia I , is given by (3.14c) and $\bar{\omega}$ stands for the square root of the average value of ω^2 . Hence

$$\bar{\omega} = (\bar{\omega^2})^{\frac{1}{2}} \quad (3.23)$$

the mass-average of ω^2 only being considered.

(I) *Behaviour of k .* Eliminating ρ_0 from (3.14a) and (3.22) we have by (3.15) and the last of the equations in (3.14e)

$$k = \frac{25 A^2}{6 M^3} \frac{\{(1 + \xi_1^2)^{\frac{1}{2}} - 1\}^2}{P(\xi_1)\{2 + \xi_1^2 + (1 + \xi_1^2)^{\frac{1}{2}}\}^2} \quad (3.24)$$

An analysis of (3.24) gives the following results when A and M remain invariable.

$$\left. \begin{aligned} L_t k &\rightarrow 0 & (a), & L_t k \xi_1 &\rightarrow 0 & (b) \\ L_t k \xi_1^2 &\rightarrow \text{finite} & (c), & L_t k &\rightarrow \frac{25}{6} \cdot \frac{A^2}{M^3} \cdot \frac{2}{\pi} \cdot \frac{1}{9} & (d) \end{aligned} \right\} \quad (3.25)$$

From the above equations it becomes clear that a and c both tend to zero when $\xi_1 \rightarrow \infty$ i.e., when the boundary becomes a sphere. The stratifications inside must then be concentric spheres, but the volume of the model becomes vanishingly small. Consequently the average density must be indefinitely large. We have, however, seen in (3.18) that $\bar{\omega^2}/2\pi\bar{\rho}$ must remain finite. Hence $\bar{\omega^2}$ also must become indefinitely large. We shall verify these consequences by actual calculations from the expressions for $\bar{\omega^2}$ and $\bar{\rho}$.

(II) *The Central Density, ρ_0 .* Eliminating k from the same set of equations as in case (I) we obtain

$$\rho_0^{\frac{1}{2}} = L \cdot \frac{P(\xi_1) \cdot (m^{\frac{2}{3}} \xi_1^3 - 1)^2}{m^{\frac{1}{3}} (m \xi_1 - 1)^{13/3}} \quad (3.26)$$

where

$$L = \frac{M^{10/3}}{A^2} \cdot \frac{2\pi(8\pi/45)^2}{(4\pi/3)^{10/3}}. \quad (3.27)$$

Hence, for $\xi_1 \gg 1$

$$\rho_0^{\frac{1}{2}} \approx L \cdot P(\xi_1) \cdot \xi_1^{\frac{1}{3}} (1 + \xi_1^2)^{2/3}, \quad \text{or} \quad \rho_0 \approx \xi_1^5 \quad (3.28a)$$

so that as $\xi_1 \rightarrow \infty$, $\rho_0 \rightarrow \infty$.

Again, for $\xi_1 \ll 1$

$$\rho_0^{\frac{1}{2}} \approx \frac{1}{\xi_1^{\frac{1}{3}}}, \quad \text{i.e.,} \quad L \rho_0 \rightarrow \infty. \quad (3.28b)$$

But $L \rho_0 \xi_1 \rightarrow$ a finite value. This last result explains why the mass remains finite

though the total volume tends to zero as $\xi_1 \rightarrow 0$ (or $e \rightarrow 1$).

The above results show that at both ends of the series ρ_0 becomes indefinitely large though at all intermediate stages it remains finite. That is, in the disc-shaped state the central density must be of a high order. Any change from this state, (where the mathematical volume is infinitesimal), leads really to an expansion in volume with an equatorial contraction. This continues till the central density attains its *minimum* value, as shown in the table below. After that the volume and the equatorial radius diminish together and ρ_0 goes on increasing. Ultimately ρ_0 reaches an order of magnitude which far exceeds the value it had at the start. The following numerical values calculated from (3.28) show the order of magnitude and the position of the minimum ρ_0 .

| ξ_1 | e | $\log \left(\frac{\rho_0^{\frac{1}{2}}}{L} \right)$ |
|---------|-------|--|
| 10 | 0.09 | 1.82264 |
| 5 | 0.196 | 0.88140 |
| 1 | 0.707 | 0.79467 |
| 0.5 | 0.89 | 1.0531 |
| 0.4 | 0.92 | 1.22424 |
| 0.3 | 0.957 | 1.59320 |

Thus, the minimum value of ρ_0 is touched in between $\xi_1 = 1$ and $\xi_1 = 0.5$, the corresponding eccentricity being nearly 0.7. The volume at this stage must be the largest attained by the model and is obviously finite.

(iii) $\bar{\omega}^2$. From (3.15) with (3.14e, iii) we have

$$\bar{\omega}^2 = 2\pi\rho_0 \frac{(1 + \xi_1^2)^{\frac{1}{2}}}{\xi_1} \cdot \frac{P(\xi_1)}{(1 + \xi_1^2)^{\frac{1}{2}} - 1}. \quad (3.29)$$

Hence

$$\bar{\omega}^2 \approx \zeta_1^3 \text{ when } \zeta_1 \gg 1, \therefore \lim_{\zeta_1 \rightarrow \infty} L_t \bar{\omega}^2 \rightarrow \infty. \quad (3.30a)$$

Also

$$\lim_{\zeta_1 \rightarrow 0} L_t \bar{\omega}^2 \rightarrow \text{a finite quantity}. \quad (3.30b)$$

Thus in the disc-shaped state the rotation is finite whereas at the other extreme, the droplet-form, the rotation is indefinitely large, the variation in the rotation in between being a monotonic increase.

6. Conclusions. From the above discussion it becomes clear that the maximum equational extension of the model is attained when $\zeta_1 \rightarrow 0$, that is, when the model turns into a flat disc. At the other end of the series it becomes a droplet of infinite concentration. And, for all variations in the surface-flattening the model plies between these two limits, attaining a finite maximum volume and a finite minimum central density when the eccentricity is near about 0.7. Such a model can never attain a state of infinite diffusion as has already been remarked in the introduction; on the contrary it can reduce itself to a point-mass.

It is interesting to compare this result with that of the model in similar stratifications (Ghosh, 1950A). It was seen that the latter could ply only between the state of infinite diffusion and the flat-disc form, attaining a finite minimum equatorial radius at the disc stage. The two series appear to meet nowhere except at the disc-state.

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SOME FORMULAS IN TENSOR CALCULUS

By

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1. In a recent paper (Sen, 1950), a finite system of 12 sets of coefficients of affine connections have been given by starting with an arbitrary set and introducing the notions of the associate and the conjugate of each coefficient of the set. In the same paper, the Christoffel symbols in the Riemannian Geometry have been obtained by special linear combinations of these coefficients. Further, some interesting relations involving these coefficients and tensors derived from these coefficients, analogous to the Riemann-Christoffel tensor, have been given. The object of this paper is to establish some formulas involving the coefficients of an arbitrary system of affine connections in combination with an arbitrary tensor and formulas involving curvature tensors derived from these coefficients in accordance with the suggestion and method given in the paper referred to above.

Let $a = \Gamma_{ij}^t$ be a set of coefficients of an arbitrary affine connection and T_{ij}^t be an arbitrary tensor. obviously

$$d = \Gamma_{ij}^t + T_{ij}^t \quad (1.1)$$

are the coefficients of another affine connection.

Also, let g_{ij} denote an arbitrary covariant symmetric tensor of rank 2 and denote the covariant derivative of g_{ij} with respect to Γ_{ij}^t by a comma followed by indices. Introducing the notions of the associate (*) and the conjugate (') of a , as given in the paper referred to above, namely

$$a^* = \Gamma_{ij}^t + g^{in} g_{in,j}, \quad a' = \Gamma_{ji}^t,$$

let

$$a_1 = a, \quad a_2 = a^*, \quad a_3 = a^{*'}, \quad a_4 = a^{**}, \quad \dots, \quad a_{12} = a'$$

and

$$d_1 = d, \quad d_2 = d^*, \quad d_3 = d^{*'}, \quad d_4 = d^{**}, \quad \dots, \quad d_{12} = d^{*'*} \dots *$$

Put

$$\alpha = g^{in} g_{in,j}, \quad \alpha_c = g^{in} g_{jn,i}, \quad \lambda = g^{in} g_{ij,n} = \lambda_c,$$

$$\beta = g^{in} g_{is} (\Gamma_{nj}^s - \Gamma_{jn}^s), \quad \beta_c = g^{in} g_{js} (\Gamma_{ni}^s - \Gamma_{in}^s), \quad \gamma = T_{ij}^t, \quad \gamma_c = T_{ji}^t,$$

$$\delta = g^{in} g_{is} T_{nj}^s, \quad \delta_c = g^{in} g_{js} T_{ni}^s, \quad \epsilon = g^{in} g_{is} T_{jn}^s, \quad \epsilon_c = g^{in} g_{js} T_{in}^s.$$

We give below the values of the finite cyclic sequence generated by d . The calculation, although laborious, is straightforward.

$$\begin{aligned} d_1 &= d = a + \gamma = a_1 + \gamma \\ d_2 &= a + \alpha - \delta = a_2 - \delta \\ d_3 &= a' + \alpha_c - \delta_c = a_3 - \delta_c \end{aligned} \quad (1.2)$$

$$\begin{aligned}
d_4 &= a + \alpha - \lambda + \beta + \epsilon_0 = a_4 + \epsilon_0 \\
d_5 &= a' + \alpha_c - \lambda + \beta_0 + \epsilon = a_5 + \epsilon_0 \\
d_6 &= a + \alpha + \alpha_0 - \lambda + \beta + \beta_0 - \gamma_0 = a_6 - \gamma_0 \\
d_7 &= a' + \alpha + \alpha_0 - \lambda + \beta + \beta_0 - \gamma = a_7 - \gamma \\
d_8 &= a' + \alpha_0 - \lambda + \beta + \beta_0 + \delta = a_8 + \delta \\
d_9 &= a + \alpha - \lambda + \beta + \beta_0 + \delta_0 = a_9 + \delta_0 \\
d_{10} &= a' + \alpha_0 + \beta_0 - \epsilon_0 = a_{10} - \epsilon_0 \\
d_{11} &= a + \alpha + \beta - \epsilon = a_{11} - \epsilon \\
d_{12} &= d' = a' + \gamma_0 = a_{12} + \gamma_0
\end{aligned}$$

If $\left\{ \begin{smallmatrix} t \\ ij \end{smallmatrix} \right\}$ be the Christoffel symbol formed with respect to g_{ij} 's, it is easily seen that for $l = 1, 2, \dots$

$$\begin{aligned}
\frac{1}{2}(d_l + d_{l+6}) &= \frac{1}{2}(a_l + a_{l+6}) = \frac{1}{2}(a + a' + \alpha + \alpha_0 - \lambda + \beta + \beta_0) \\
&= \frac{1}{2}[\Gamma_{ij}^l + \Gamma_{ji}^l + g^{tn}(g_{in,j} + g_{jn,i} - g_{ij,n}) + g^{tn}\{g_{is}(\Gamma_{nj}^s - \Gamma_{jn}^s) + g_{is}(\Gamma_{ni}^s - \Gamma_{in}^s)\}] \\
&= \frac{1}{2}g^{tn}\left\{\frac{\partial g_{in}}{\partial x^j} + \frac{\partial g_{jn}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^n}\right\} = \left\{ \begin{smallmatrix} t \\ ij \end{smallmatrix} \right\} \quad (1.3)
\end{aligned}$$

Further

$$g_{is}(\Gamma_{ij}^t + T_{ij}^t) + g_{is}(\Gamma_{ji}^t + g^{tn}g_{in,j} - g^{tn}g_{jn,i}) = g_{is,j} + g_{is}\Gamma_{ij}^t + g_{is}\Gamma_{ji}^t = \frac{\partial g_{is}}{\partial x^j} \quad (1.4)$$

as are to be expected.

Now, as a special case, let

$$d = a + m(b - c) = W_{ij}^t, \text{ say,} \quad (1.5)$$

where $b = L_{ij}^t$ and $c = \Omega_{ij}^t$ are sets of coefficients of two arbitrary affine connections and m is a numerical constant.

For the sake of convenience, put $b - c = T_{ij}^t$. Let the covariant derivatives with respect to b, c, d be denoted by a semi-colon, a solidus and a square bracket followed by indices.

We have then

$$[g_{in}]_j = \frac{\partial g_{in}}{\partial x^j} - g_{in}W_{ij}^s - g_{is}W_{nj}^s = g_{in,j} - m\{g_{in}T_{ij}^s + g_{is}T_{nj}^s\}.$$

Therefore

$$g^{tn}[g_{in}]_j = g^{tn}g_{in,j} - mT_{ij}^t - mg^{tn}g_{is}T_{nj}^s.$$

Hence

$$\begin{aligned}
W_{ij}^* &= W_{ij}^t + g^{tn}[g_{in}]_j = \Gamma_{ij}^t + mT_{ij}^t + g^{tn}g_{in,j} - mT_{ij}^t - mg^{tn}g_{is}T_{nj}^s \\
&= (\Gamma_{ij}^t + g^{tn}g_{in,j}) + mg^{tn}\{g_{in,j} - g_{in,j} + g_{is}T_{nj}^s\} = \Gamma_{ij}^{*t} + m\{L_{ij}^{*t} - \Omega_{ij}^{*t}\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
d^* &= a^* + m(b^* - c^*) \\
d' &= a' + m(b' - c') \quad \left\{ \right. \quad (1.6)
\end{aligned}$$

The values of d_l can now explicitly be given in terms of a_l, b_l and $m, l = 1, \dots, 12$.

In particular, let $d = u + m(a_1 - a_7)$, where $u = \left\{ \begin{smallmatrix} t \\ ij \end{smallmatrix} \right\}$.

We may now write

$$d_1 = \frac{1}{2}(d_1 + d_7) + m(a_1 - a_7), \text{ or } \frac{1}{2}(d_1 - d_7) = m(a_1 - a_7).$$

Therefore the values (1.2) are now replaced by (1.2'), say, where a and a' are replaced by u , T_{η}^t is replaced by $\frac{1}{2}(d_1 - d_7)$ and $\alpha, \alpha_e, \lambda, \beta, \beta_e$ have the values zero. Further, since

$$a_1 + a_7 = 2u, \text{ or } a_1 - a_7 = 2(a_1 - u),$$

we have

$$d - u = 2m(a_1 - u), \text{ or } d = 2ma_1 - (2m - 1)u.$$

Therefore

$$d = \frac{1}{2}(a + u) + \frac{4m - 1}{2}(a - u).$$

Hence, in order to obtain the values of d_1, \dots, d_{12} , we may also replace (1.2) by (1.2'), say, where a, a' are replaced by $\frac{1}{2}(a + u), \frac{1}{2}(a' + u)$ respectively. T_{η}^t is replaced by $\frac{4m - 1}{2}(a - u)$ and $\alpha, \alpha_e, \lambda, \beta, \beta_e$ are replaced respectively by half their values.

2. Let us now take as a further special case

$$d = \frac{1}{2}(a + b) + m(a - b) \quad (2.1)$$

Put $\frac{1}{2}(a + b) = \Lambda_{\eta}^t$ and $a - b = T_{\eta}^t$ and let the covariant derivatives of g_{ij} 's with respect to Λ_{η}^t be denoted by an ordinary bracket followed by indices. Also let

$$C(a) = \Gamma_{\eta k}^t = \frac{\partial \Gamma_{ik}^t}{\partial x^j} - \frac{\partial \Gamma_{ij}^t}{\partial x^k} + \Gamma_{ij}^s \Gamma_{sk}^t - \Gamma_{ik}^s \Gamma_{sj}^t$$

be the curvature tensor formed with respect to a . Similarly for $C(b), C(c)$ etc. We, then, have

$$C(b) - C(a) = T_{\eta, k}^t - T_{ik, j}^t + T_{ij}^t T_{ik}^s - \Gamma_{sk}^t T_{ij}^s + T_{is}^t (\Gamma_{jk}^s - \Gamma_{kj}^s) \quad (2.2)$$

and

$$\frac{1}{2}\{C(a) + C(b)\} - C(\frac{1}{2}(a + b)) = \frac{1}{2}(T_{\eta}^t T_{ik}^s - T_{ik}^t T_{\eta}^s). \quad (2.3)$$

Subtracting (2.3) from (2.2)

$$4C(\frac{1}{2}(a + b)) - \{3C(a) + C(b)\} = T_{\eta, k}^t - T_{ik, j}^t + T_{is}^t (\Gamma_{jk}^s - \Gamma_{kj}^s). \quad (2.4)$$

Interchanging a, b in (2.4) and subtracting

$$C(b) - C(a) = (T_{\eta}^t)_k - (T_{ik})_j + T_{is}^t (\Lambda_{jk}^s - \Lambda_{kj}^s). \quad (2.5)$$

Put $\frac{1}{2}(a + b)$ for a and $\frac{1}{2}(a + b) + m(b - a)$ for b in (2.4), so that $a - b$ is replaced by $m(a - b)$ and $\frac{1}{2}(a + b)$ is replaced by $\frac{1}{2}(a + b) + \frac{1}{2}m(b - a)$. Hence (2.4) reduces to

$$\begin{aligned} 4C(\frac{1}{2}(a + b) + \frac{1}{2}m(b - a)) - \{3C(\frac{1}{2}(a + b) + C(\frac{1}{2}(a + b) + m(b - a)))\} \\ = m[(T_{\eta}^t)_k - (T_{ik})_j + T_{is}^t (\Lambda_{jk}^s - \Lambda_{kj}^s)] \end{aligned} \quad (2.6)$$

From (2.5) and (2.6)

$$m\{C(b) - C(a)\} = 4C(\frac{1}{2}(a + b) + \frac{1}{2}m(b - a)) - \{3C(\frac{1}{2}(a + b)) + C(\frac{1}{2}(a + b) + m(b - a))\}$$

interchanging a, b and subtracting

$$\begin{aligned} 2m\{C(a) - C(b)\} = 4\{C(\frac{1}{2}(a + b) + \frac{1}{2}m(a - b)) - C(\frac{1}{2}(a + b) + \frac{1}{2}m(b - a))\} \\ - \{C(\frac{1}{2}(a + b) + m(a - b)) - C(\frac{1}{2}(a + b) + m(b - a))\} \end{aligned} \quad (2.7)$$

If instead of subtracting we add, we get

$$4\{C(\tfrac{1}{2}(a+b) + \tfrac{1}{2}m(a-b)) + C(\tfrac{1}{2}(a+b) + \tfrac{1}{2}m(b-a))\} \\ = 8C(\tfrac{1}{2}(a+b)) + \{C(\tfrac{1}{2}(a+b) + m(a-b)) + C(\tfrac{1}{2}(a+b) + m(b-a))\}. \quad (2.8)$$

Further, it follows directly from (2.8) that

$$m^2[C(\tfrac{1}{2}(a+b)) - \tfrac{1}{2}\{C(a) + C(b)\}] = C(\tfrac{1}{2}(a+b)) - \tfrac{1}{2}\{C(\tfrac{1}{2}(a+b) + \tfrac{1}{2}m(a-b)) \\ + C(\tfrac{1}{2}(a+b) + \tfrac{1}{2}m(b-a))\}. \quad (2.9)$$

The last three results appear interesting of which the last two reduce to one given in the paper referred to before for the particular cases when $m = 1$ and $m = 2$ respectively. Other formulas can be derived from them by taking the associates and the conjugates. For example, $a_i - b_k$ can always be replaced by $b_{k+s} - a_{i+s}$. Again let

$$e_1 = (a, b) = \tfrac{1}{2}(a+b) + \tfrac{1}{2}m(a-b), \quad \bar{e}_1 = (b, a) = \tfrac{1}{2}(a+b) + \tfrac{1}{2}m(b-a).$$

Then

$$e_2 = (e_1, \bar{e}_1) = \tfrac{1}{2}(a+b) + \tfrac{1}{2}m^2(a-b), \quad \bar{e}_2 = (\bar{e}_1, e_1) = \tfrac{1}{2}(a+b) + \tfrac{1}{2}m^2(b-a).$$

Similarly for e_3, e_4 etc. In general,

$$e_r = \tfrac{1}{2}(a+b) + \tfrac{1}{2}m^r(a-b) \quad \text{and} \quad \bar{e}_r = \tfrac{1}{2}(a+b) + \tfrac{1}{2}m^r(b-a), \quad r = 1, 2, 3, \dots$$

Also let

$$C[a, b] = \tfrac{1}{2}\{C(a) + C(b)\} - C(\tfrac{1}{2}(a+b)) = T_{ik}^t, \text{ say}$$

Then it follows from (2.8) that

$$C[e_r, \bar{e}_r] = m^{2r}T_{ik}^t, \quad r = 1, 2, 3, \dots$$

We may now obtain various formulae by eliminating the tensor T_{ik}^t . For example,

$$C[e_p, \bar{e}_p].C[e_q, \bar{e}_q] = C[a, b].C[e_{p+q}, \bar{e}_{p+q}]. \quad (2.10)$$

Other formulae can also be derived by taking the associates and the conjugates. For $C[a_i, b_k]$ can always be replaced by $C[a_{i+s}, b_{k+s}]$.

In conclusion, I take the opportunity of acknowledging my grateful thanks to Dr. R. N. Sen for his helpful guidance and encouragement.

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STRESSES DUE TO NUCLEI OF THERMO-ELASTIC STRAIN IN A THIN CIRCULAR PLATE.

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INTRODUCTION.

It has been shown by Goodier (1937) that if a part of an elastic plate be heated while the remaining part is kept at zero temperature, the displacement can be expressed as the gradient of a certain potential function. This result has been utilized in this paper to find the plane stresses in an isotropic circular plate of small thickness when a small element of area surrounding an arbitrary point in the plate is heated in the manner stated above. The method developed by the author in a previous paper (Sen, 1946) has been used to solve the problem.

SOLUTION

Let O , the centre of a plane face of the thin circular plate be the origin and Ox, Oy , two perpendicular lines on this face, the axes of co-ordinates. We suppose that an element of area $d\Omega$ surrounding the point $A (c, o)$ is heated to a temperature T , the remaining part being at zero temperature. If r_1 be the distance of any point (x, y) from A , then following the arguments given by Goodier (1937) we find that the components of displacement (u, v) at the point (x, y) can be expressed as

$$u = \frac{\partial \varphi}{\partial x}, v = \frac{\partial \varphi}{\partial y}, \quad (1.1)$$

where

$$\varphi = \frac{\alpha T}{2\pi} (1 + \sigma) d\Omega \log r_1, \quad (1.2)$$

α being the coefficient of linear expansion, and σ , Poisson's ratio. Plane stresses produced by these components of displacement are

$$\begin{aligned} \widehat{xx}_1 &= \frac{E\alpha T d\Omega}{2\pi} \left[\frac{1}{r_1^2} - \frac{2(x-c)^2}{r_1^4} \right], \\ \widehat{yy}_1 &= \frac{E\alpha T d\Omega}{2\pi} \left[\frac{1}{r_1^2} - \frac{2y^2}{r_1^4} \right], \\ \widehat{xy}_1 &= -\frac{E\alpha T d\Omega}{\pi} \frac{(x-c)y}{r_1^4}, \end{aligned} \quad (1.3)$$

in which E is Young's modulus, and $r_1^2 = (x-c)^2 + y^2$. Putting Q for $E\alpha T d\Omega/2\pi$, and $r^2 = x^2 + y^2$, we obtain the expressions

$$\begin{aligned} r.\widehat{rx}_1 &= x.\widehat{xx}_1 + y.\widehat{xy}_1 = \frac{Q}{r_1^4} [2cr^2 - x(r^2 + c^2)], \\ r.\widehat{ry}_1 &= x.\widehat{xy}_1 + y.\widehat{yy}_1 = \frac{Qy}{r_1^4} [c^2 - r^2]. \end{aligned} \quad (1.4)$$

If P be a point (x, y) on the circular boundary $r = a$, and $B(a^2/c, 0)$ the inverse point of A with respect to the circle, we have

$$\begin{aligned} [r.\widehat{rx}_1]_{r=a} &= \frac{Q}{AP^4} [2ca^2 - x(a^2 + c^2)] = \frac{Q a^4}{BP^4 \cdot c^4} [2ca^2 - x(a^2 + c^2)], \\ [r.\widehat{ry}_1]_{r=a} &= \frac{Qy}{AP^4} [c^2 - a^2] = \frac{Q \cdot a^4 y}{BP^4 \cdot c^4} [c^2 - a^2]. \end{aligned} \quad (1.5)$$

To nullify the above stresses on the circular boundary we should superpose a stress system $\widehat{xx}_2, \widehat{yy}_2, \widehat{xy}_2$ which yield

$$\begin{aligned} r.\widehat{rx}_2 &= x.\widehat{xx}_2 + y.\widehat{xy}_2, \\ r.\widehat{ry}_2 &= x.\widehat{xy}_2 + y.\widehat{yy}_2 \end{aligned}$$

such that

$$[r.\widehat{rx}_2]_{r=a} = \frac{Q \cdot a^4}{BP^4 \cdot c^4} [x(a^2 + c^2) - 2ca^2],$$

$$\text{and} \quad [r.\widehat{ry}_2]_{r=a} = \frac{Q \cdot a^4 y}{BP^4 \cdot c^4} [a^2 - c^2]. \quad (1.6)$$

It has been shown in the author's paper (Sen, 1946) that if the boundary values of the expressions $r.\widehat{rx}_2$ and $r.\widehat{ry}_2$ be known in the forms $a[L(z)]_{r=a}$ and $a[M(z)]_{r=a}$ where $z = x + iy$ and $L(z), M(z)$ are analytic functions of z , then

$$\begin{aligned} r.\widehat{rx}_2 &= R \left[\frac{r^2 - a^2}{4} \left\{ \frac{f(z) - z f'(z)}{z} \right\} + aL(z) \right], \\ r.\widehat{ry}_2 &= R \left[\frac{r^2 - a^2}{4} \left\{ \frac{f(z) - z f'(z)}{z} \right\} + aM(z) \right]. \end{aligned} \quad (1.7)$$

In these expressions R denotes the real part, and $f(z)$ is an analytic function of z such that

$$\widehat{xx}_2 + \widehat{yy}_2 = R[f(z)]. \quad (1.8)$$

It was also proved in the same paper that

$$f(z) = 2a \frac{L(z) + iM(z)}{z} \quad (1.9)$$

except when

$$f(z) \propto \frac{1}{z}.$$

It is evident from (1.6) that we can put

$$aL(z) = \frac{Qa^2}{c^2} \cdot \frac{z}{(z-a^2/c)^2},$$

$$aM(z) = -\frac{Qa^2}{c^2} \cdot \frac{iz}{(z-a^2/c)^2}. \quad (1.10)$$

Hence from (1.9) we get

$$f(z) = \frac{4Qa^2}{c^2(z-a^2/c)^2} \quad (1.11)$$

Thus we find that the expressions $r.\widehat{rx}_2$, $r.\widehat{ry}_2$ (and hence the superposed stress system) can be completely determined from (1.7). These combined with \widehat{xx}_1 , \widehat{xy}_1 , \widehat{yy}_1 , given in (1.3) give us the required stress distribution in the plate.

The hoop stress $\widehat{\theta\theta}$ at any point on the boundary can be found out from the boundary value of $\odot (= \widehat{rr} + \widehat{\theta\theta})$. The contribution to \odot by \widehat{xx}_1 and \widehat{yy}_1 is nil. Since at the circular edge $\widehat{rr} = 0$,

$$\begin{aligned} [\widehat{\theta\theta}]_{r=a} &= [\odot]_{r=a} = [Rf(z)]_{r=a} \\ &= \left[R \left\{ \frac{4Qa^2}{c^2(z-a^2/c)^2} \right\} \right]_{r=a} \\ &= \frac{-2E\alpha T d\Omega}{\pi} \cdot \frac{a^2 - 2ac \cos \theta + c^2 \cos 2\theta}{[a^2 - 2ac \cos \theta + c^2]^2}, \end{aligned} \quad (1.12)$$

where (a, θ) are the co-ordinates of the point on the boundary.

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